Fixed Points and Noetherian Topologies

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What to bring home from this talk?

♣ Old things

- what a well-quasi-order is.
- what Noetherian rings and Noetherian spaces are.
- how to inductively define well-quasi-orders.

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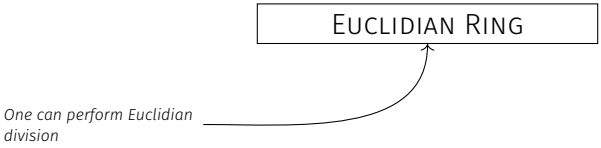
♦ New things

- how to inductively define Noetherian topologies.
- “simpler” proofs of previous constructions.
- justification that previous definitions were “good”.

Thank you!

Wait ...

What is a Noetherian space??


Euclidian Ring
One can perform Euclidian division

$(\mathbb{R}[X_1, \ldots, X_n], +, \times)$

$(\mathbb{C}[X, Y], +, \times)$

$(\mathbb{Q}[\sqrt{-19}], +, \times)$

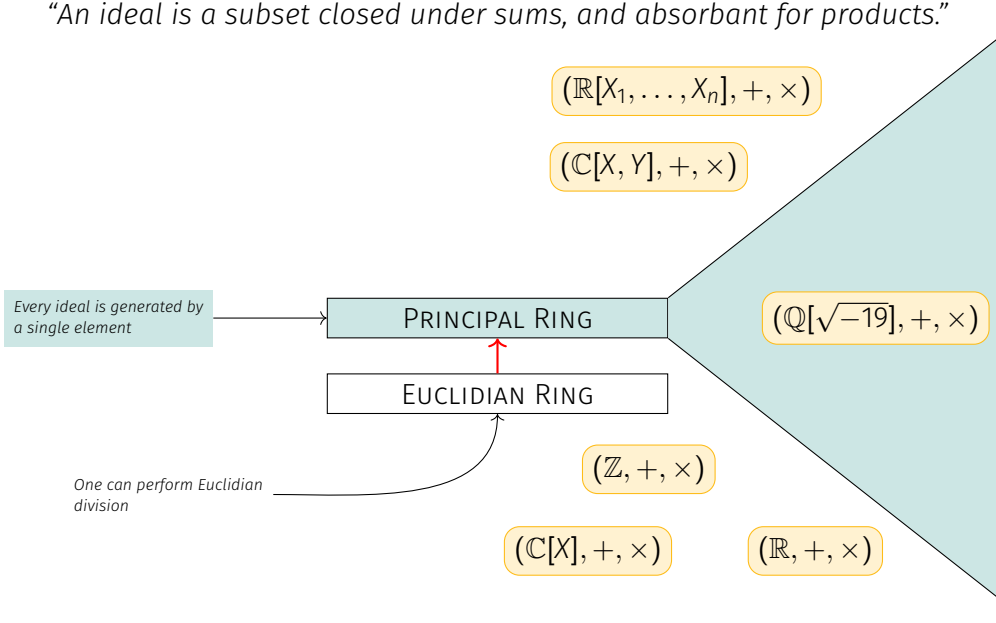
Euclidian Ring

One can perform Euclidian division

$(\mathbb{Z}, +, \times)$

$(\mathbb{C}[X], +, \times)$ $(\mathbb{R}, +, \times)$

"An ideal is a subset closed under sums, and absorbant for products."



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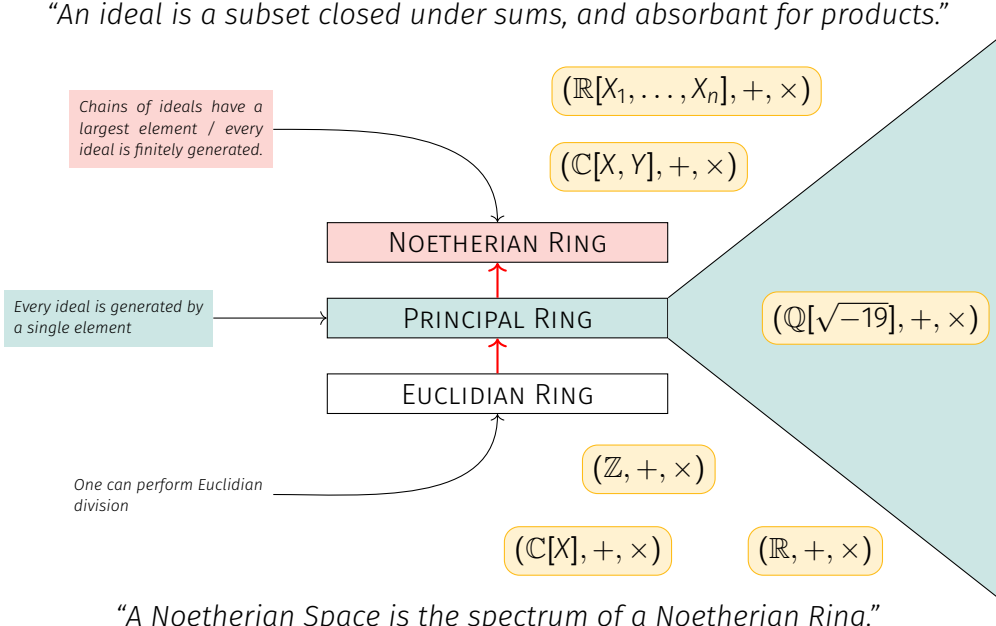
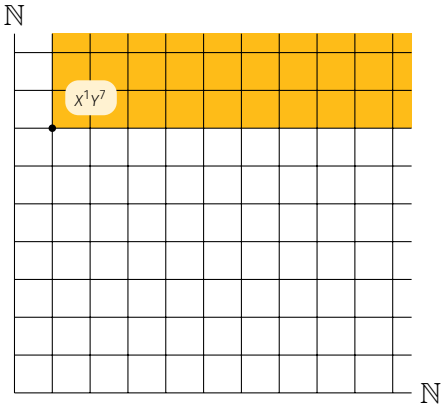


"A Noetherian Space is the spectrum of a Noetherian Ring."

Looking at ideals generated by $X^n Y^m$.



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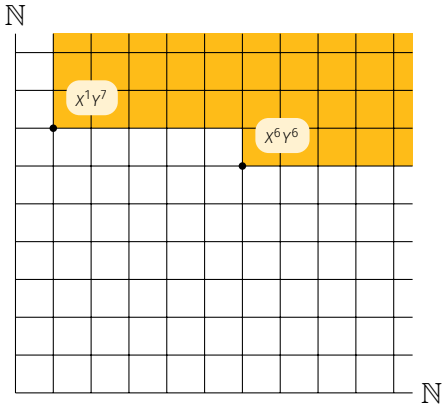

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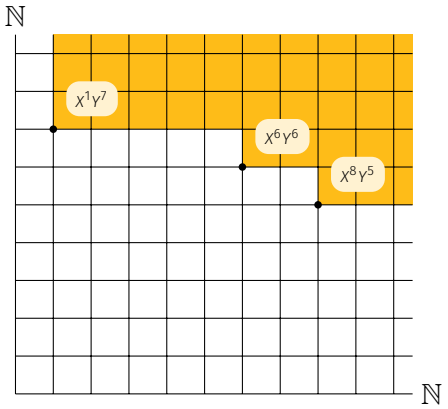

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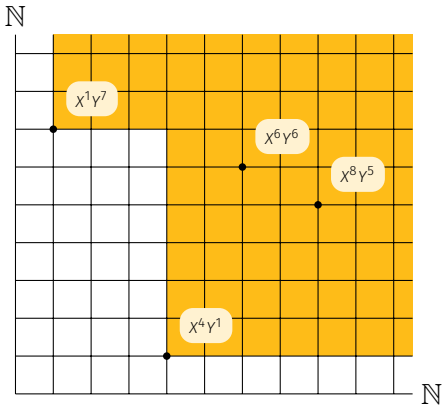

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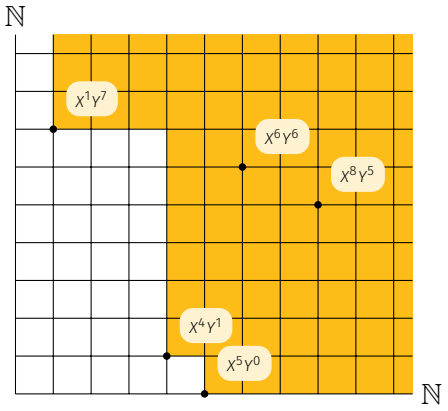

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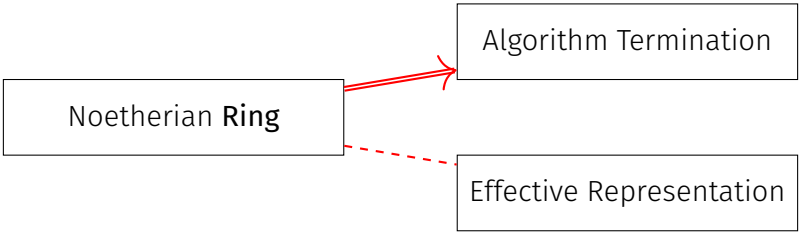
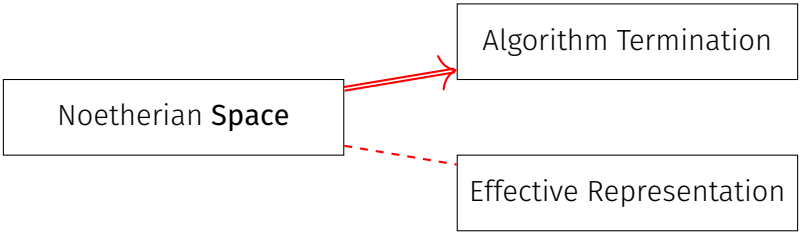


"Chains of ideals stabilize"

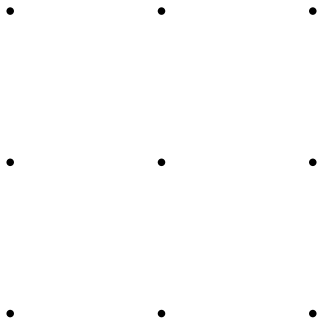


"Chains of open properties stabilize"

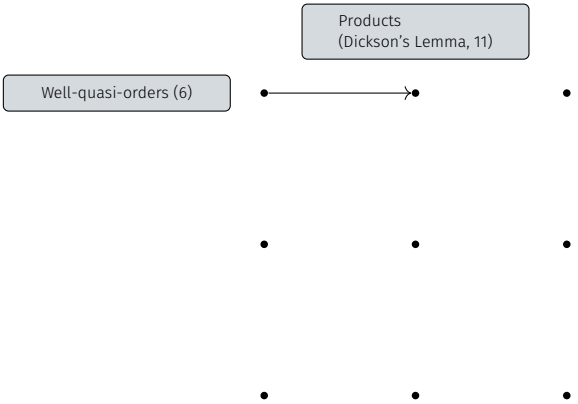
Wait ...

What's the issue again?

Well-quasi-orders (6)



A little story



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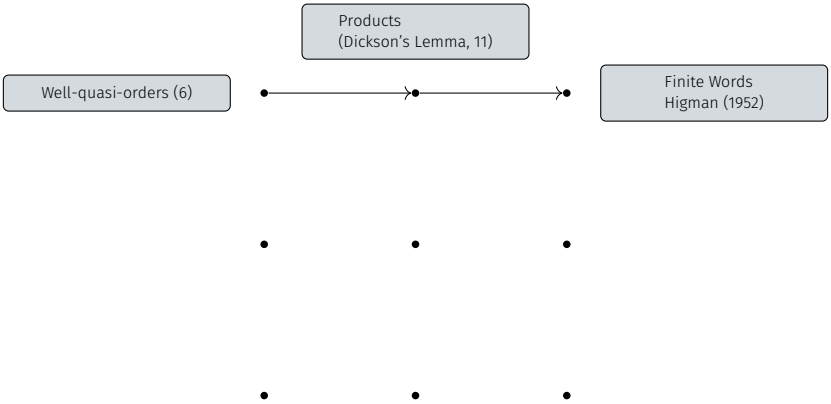

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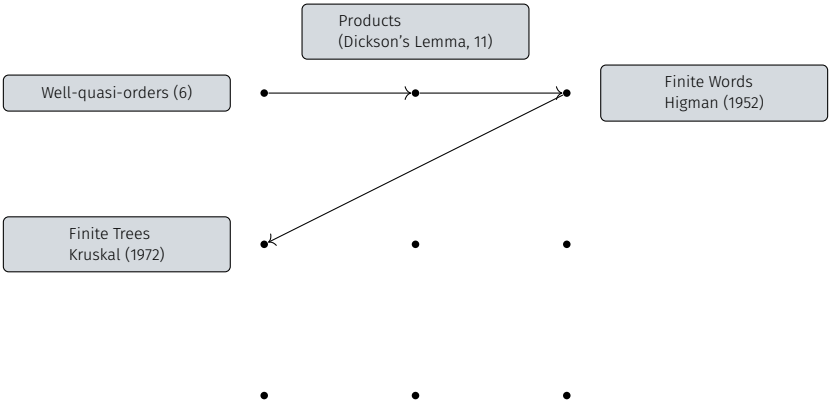

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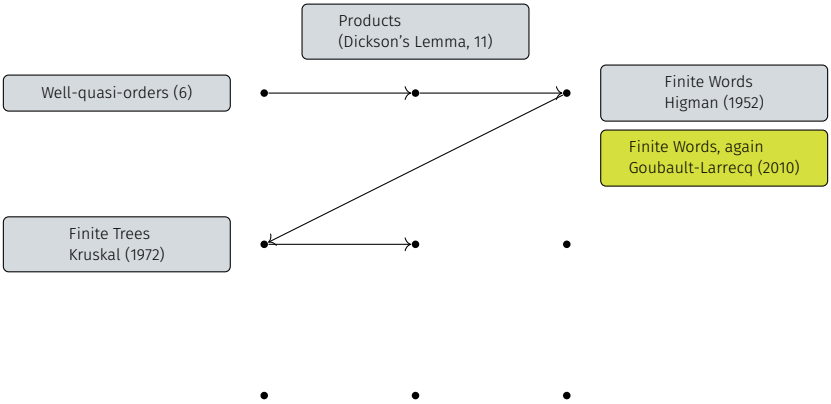

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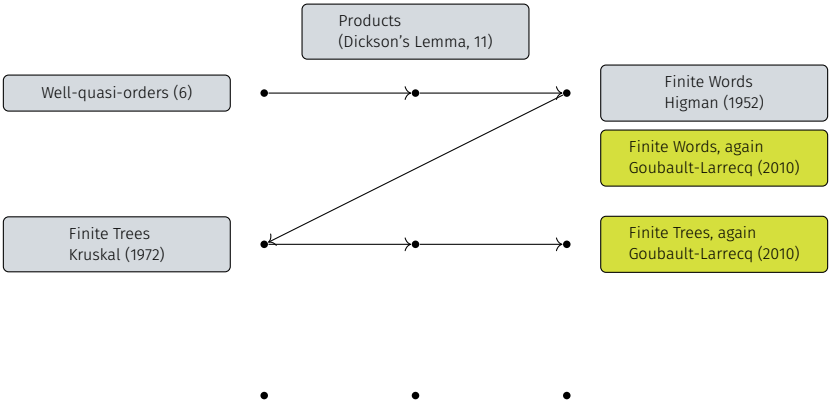

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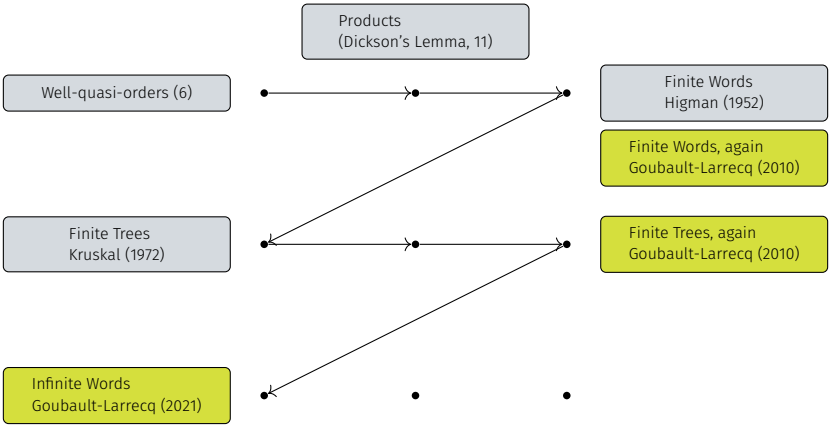

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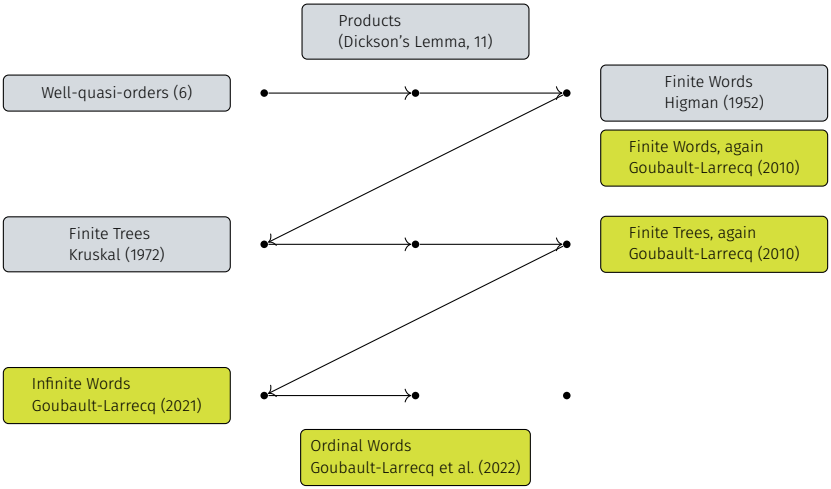

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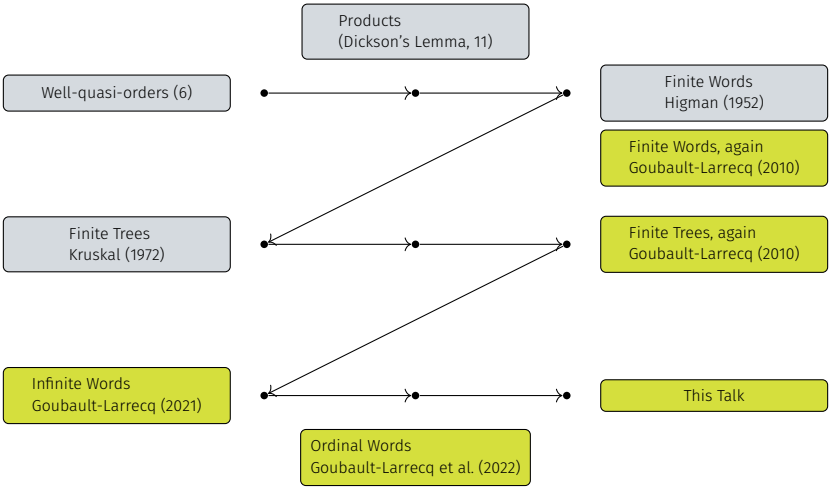

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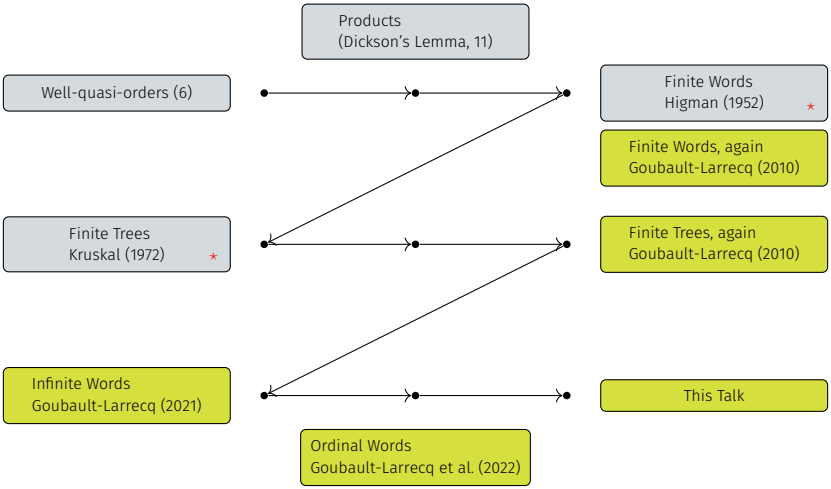
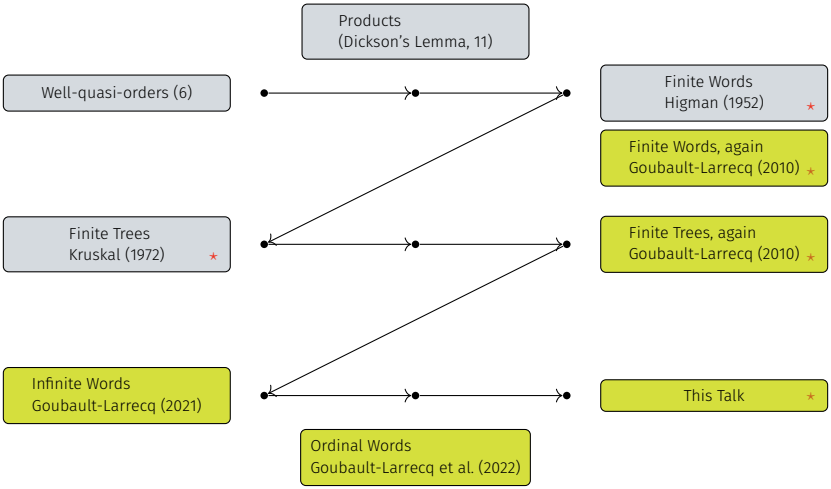


⋆Minimal bad sequence arguments, Nash-Williams (1965)

A little story



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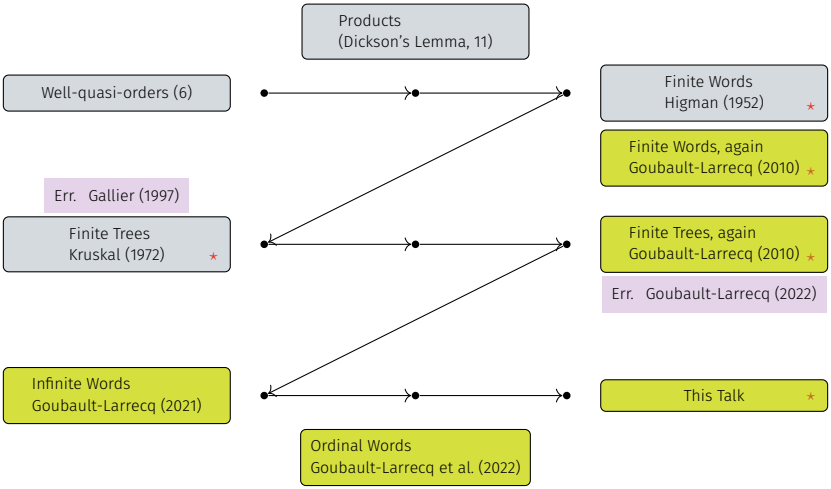

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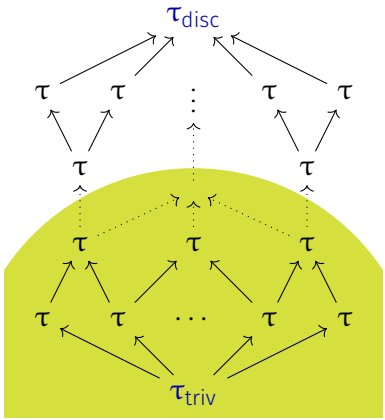
Two questions

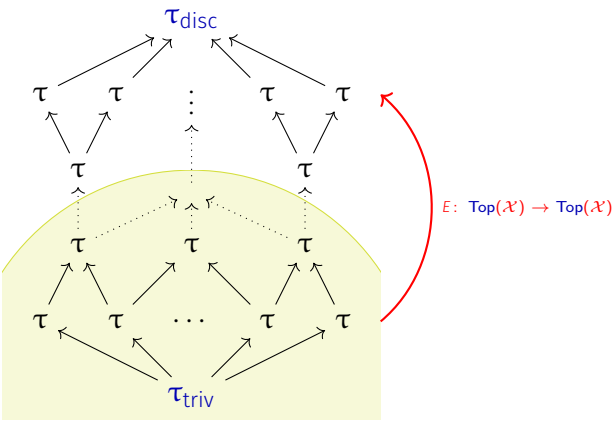
1. Why these topologies/orderings?
2. Why not a generic theorem?

A good question ...



Contributions

A fixed point theorem ... "as promised"



With E monotone and fixing Noetherian topologies.

Theorem

If E is monotone, fixes Noetherian topologies, and respects subsets, then the least fixed point of E is a Noetherian topology.

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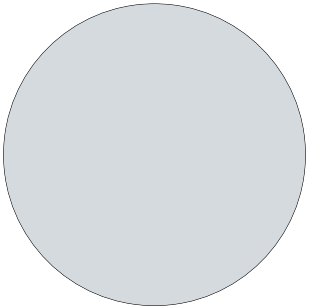
♣ Remarks

- The extra condition is needed
- The proof uses a topological minimal bad sequence argument

For all τ, H closed subset of τ,

$$E(\tau \mid H) \mid H = E(\tau) \mid H$$



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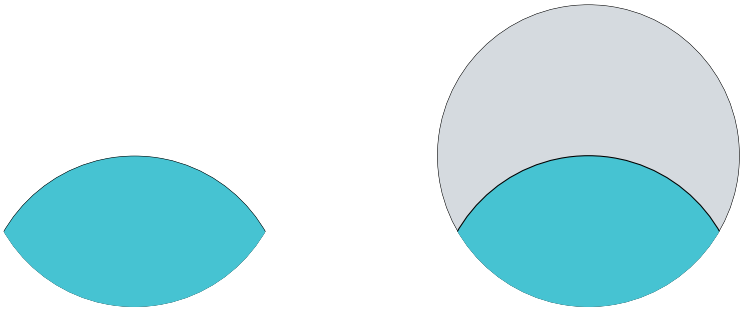

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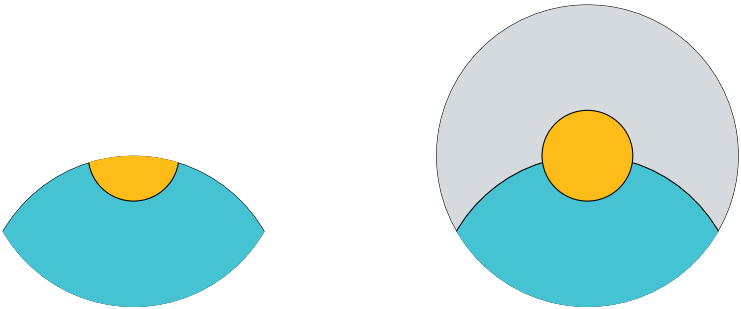

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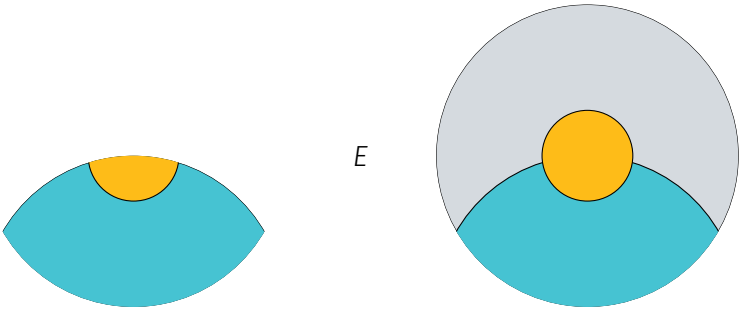

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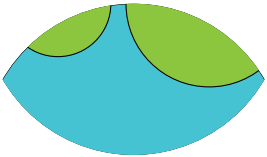

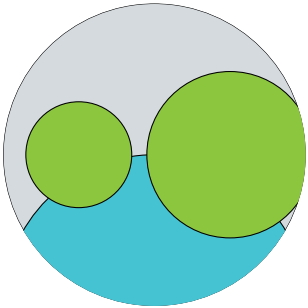

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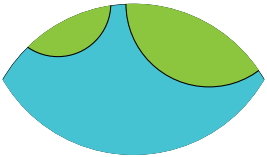

Contributions

How does it answer anything?

Theorem

Given an inductively defined space $\mathcal{X} = F(\mathcal{X})$, one can derive a generic topology expander.

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Given an inductively defined space $\mathcal{X} = F(\mathcal{X})$, one can derive a generic topology expander.

♣ Remarks

- Gives back the previous topologies for finite words and finite trees!
- Correctly generalizes with what is done in the realm of well-quasi-orders, e.g., by Hasegawa (2002).

But we can do better

♣ Non inductive cases...

- Infinite words,
- Ordinal words,
- Ordinal branching trees.

Conclusion

What's to remember now?

Outlook

♦ Takeaway

- You can build reasonable inductive topologies
- You can forget analytic functors

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