CAMBRIDGE
UNIVERSITY PRESS

ARTICLE

Measuring well-quasi-ordered finitary powersets

Sergio Abriola, Simon Halfon, Aliaume Lopez, Sylvain Schmitz, Philippe Schnoebelen, and Isa Vialard

Universidad de Buenos Aires, ICC CONICET, Buenos Aires, Argentina
Université Paris-Saclay, CNRS, ENS Paris-Saclay, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France
Université Paris Cité, CNRS, IRIF, F-75013, Paris, France
Université Paris-Saclay, CNRS, ENS Paris-Saclay, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France
alopez@irif.fr
Université Paris Cité, CNRS, IRIF, F-75013, Paris, France
Université Paris-Saclay, CNRS, ENS Paris-Saclay, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France
Université Paris-Saclay, CNRS, ENS Paris-Saclay, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France

Abstract

The complexity of a well-quasi-order (wqo) can be measured through three ordinal invariants: the width as a measure of antichains, height as a measure of chains, and maximal order type as a measure of bad sequences.

We study these ordinal invariants for the finitary powerset, i.e., the collection $P_f(A)$ of finite subsets of a wqo A ordered with the Hoare embedding relation. We show that the invariants of $P_f(A)$ cannot be expressed as a function of the invariants of A, and provide tight upper and lower bounds for them.

We then focus on a family of well-behaved wqos, for which these invariants can be computed compositionally, using a newly defined ordinal invariant called the *approximate* maximal order type. This family is built from multiplicatively indecomposable ordinals, using classical operations such as disjoint unions, products, finite words, finite multisets, and the finitary powerset construction.

Keywords: Well-quasi-orders; ordinal measures; powersets; descriptive complexity.

1. Introduction

Well-quasi-orders (wqos) are an extensively used and studied tool in mathematics, logic and computer science (see, e.g., the recent book by **?**), with a wide range of applications, notably in graph theory (Robertson and Seymour, 2004) and in our own specific field of interest, the automatic verification of computer programs and systems (Dershowitz and Manna, 1979; Abdulla et al., 2000; Finkel and Schnoebelen, 2001; Blass and Gurevich, 2008; **?**).

For a finite partial order (which is always a wqo), several natural invariants exist: its size, its "height", and its "width", as illustrated in Figure 1. Analogues of these measures also exist for general (i.e. infinite) well-quasi-orders. The *maximal order type* (or m.o.t.) was originally defined by de Jongh and Parikh (1977) as the order type of the maximal linearisation of a wqo. Schmidt (1981) then introduced the *ordinal height* as the order type of a maximal chain of a wqo. Later Kříž and Thomas (1990) crafted the notion of *ordinal width*, standardized the definitions of these three *ordinal invariants*, and proved numerical relationships between them (see, for instance, Theorem 2.12 and its corollary).

For well-structured transition systems, i.e., computational systems that rely on an underlying wqo (Abdulla et al., 2000; Finkel and Schnoebelen, 2001; Bonnet et al., 2013), one can assign complexity upper bounds related to maximal order types of wqos through the technology of *length functions theorems* and *controlled bad sequences* (e.g., Figueira et al., 2011; Abriola et al., 2015;
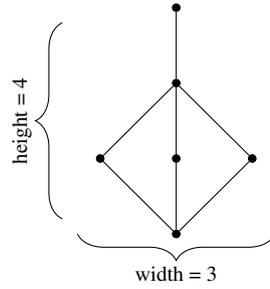
Figure 1: A partial order of size 6, height 4 and width 3, described by its Hasse diagram.

Balasubramanian, 2020). Schmitz (2019) refined this technique by using *controlled antichains* to prove complexity upper bounds related to widths of wqos instead of maximal order types. In all these examples, the computational complexity bounds are derived from bounds on the ordinal invariants of the underlying wqos.

When, as is often the case in programming, a wqo is obtained by combining simpler wqos, computing its ordinal invariants can often be done compositionally since usually these invariants are expressible as a function of the invariants of the components. De Jongh and Parikh initiated this line of thinking by computing the m.o.t. of disjoint sums and Cartesian products of wqos (de Jongh and Parikh, 1977), followed by Schmidt with the m.o.t. of word embedding and homeomorphic tree embedding on a wqo (Schmidt, 2020) and Van der Meeren et al. with the m.o.t. of the finite multiset construction (Van der Meeren et al., 2015; Weiermann, 2009). Considering the other invariants, Abraham and Bonnet measured the height of Cartesian products, and also the width of disjoint sums and lexicographic products (Abraham and Bonnet, 1999). We refer to (Džamonja et al., 2020) for a recent survey of these questions.

However, there are useful operations on wqos for which computing the ordinal invariants remains a challenge. For instance, the width of the Cartesian product $A \times B$ of two wqos is not a function of the ordinal invariants of A and B, as shown by Vialard who tackled that issue by exhibiting a family of *elementary* wqos, for which the width of Cartesian products is compositionally computable (Vialard, 2024b). Similarly, the width of the multiset ordering with elements from A cannot be expressed as a function of A's width, height and m.o.t., and Vialard (2023) proved it is equal to $\omega^{\mathbf{o}_\perp(A)}$, introducing a new ordinal invariant, $\mathbf{o}_\perp(A)$, the *friendly order type* of A.

This article focuses on another difficult case, namely the finitary powerset construction, i.e., the set $\mathsf{P}_\mathsf{f}(A)$ of all *finite* subsets of some wqo A, ordered by embedding (unlike the full powerset, this is a wqo whenever A is a wqo).

Contributions. Our contributions are threefold. First, we establish lower and upper bounds on the invariants of $\mathsf{P}_\mathsf{f}(A)$ with respect to those of A, as summarized in Table 1. Second, we prove that these bounds are tight by constructing two families of wqos, $\mathscr{P}$ and $\mathscr{H}$ reaching those bounds. Finally, noticing that the examples witnessing the tightness of the bounds satisfy the peculiar property that $\mathbf{o}(A) = \mathbf{w}(A)$, and because our bounds prove that $\mathbf{o}(\mathsf{P}_\mathsf{f}(A)) = \mathbf{w}(\mathsf{P}_\mathsf{f}(A))$ in such cases, we investigate a specific family of wqos built inductively as described in Definition 5.1 that we call *elementary* wqos. While the ordinal invariants of $\mathsf{P}_\mathsf{f}(A)$ remain non-functional with respect to those of A for elementary wqos, we showed that by introducing *weakened* versions of the usual ordinal invariants (see Definition 5.8), one could recover compositional computability of weakened and non-weakened invariants over elementary wqos, as shown in Section 5. Namely, we provide in Tables 5 and 6 (respectively in page 23 and page 27) recursive equations to compute the ordinal invariants of elementary wqos.

Table 1. : Tight upper and lower bounds for the ordinal invariants of $\mathsf{P}_\mathsf{f}(A)$.

Invariant	Lower Bound	Upper Bound	
$\mathbf{o}(\mathsf{P}_\mathsf{f}(A))$	$1 + \mathbf{o}(A)$	$2^{\mathbf{o}(A)}$	Theorems 3.3 and 4.3
$\mathbf{h}(\mathsf{P}_\mathsf{f}(A))$	$1 + \mathbf{h}(A)$	$2^{\mathbf{h}(A)}$	Theorem 3.6
$\mathbf{w}(\mathsf{P}_\mathsf{f}(A))$	$2^{\mathbf{w}(A)}$	none	Theorems 3.11 and 4.3

Outline of the article. Section 2 recalls the definitions that will be used in this article. Section 3 establishes upper and lower bounds for the ordinal invariants on $\mathsf{P}_\mathsf{f}(A)$. Section 4 provides families of wqos attaining those bounds. Finally, Section 5 shows that restricting ourselves to elementary wqos, we can compute the ordinal invariants and their weakened version compositionally.

Genesis of this article. This article grew from unpublished notes (2013) by Abriola, Schmitz and Schnoebelen who investigated the m.o.t. of $\mathsf{P}_\mathsf{f}(A)$, established the upper and lower bounds we give in Section 3.1, and proved their tightness. The observation that $\mathbf{o}(\mathsf{P}_\mathsf{f}(A))$ is not a function of $\mathbf{o}(A)$ spurred research in the width invariant, with the hope that $\mathbf{o}(\mathsf{P}_\mathsf{f}(A))$ could be better characterised when both $\mathbf{o}(A)$ and $\mathbf{w}(A)$ are known but this idea proved inconclusive. Then, in 2020, Halfon and Lopez revived this line of work and gave bounds for the height and width of $\mathsf{P}_\mathsf{f}(A)$, together with proofs of tightness. In 2022, Vialard joined the group and showed how one can determine exactly the values of all three invariants for the rather large family of wqos given in Section 5. It was then decided by all six contributors that this material will profit from being collected in a single text.

2. Basics

We assume that the reader is familiar with basic notions on ordinals and quasi-orders. We recall here the definitions and notations that will be used throughout the article.

Well-quasi-orderings. A sequence $(x_i)_{i \in \mathbb{N}}$ is *good* in a quasi-order $(A, \leq)$ when there exists an increasing pair $x_i \leq x_j$ with $i < j$. Otherwise it is *bad*. When the ordering can be inferred from the context, we may write just A instead of $(A, \leq)$ for a quasi-order. A *well-quasi-order* (or wqo) is a quasi-order in which there are no infinite bad sequences. A wqo can alternatively be defined as a quasi-order which is *well-founded* (it has no infinite decreasing sequences) and satisfies the *finite antichain condition* (it has no infinite *antichains*: sets of pairwise incomparable elements).[a]

If a wqo $(A, \leq)$ is such that $\leq$ is anti-symmetric, then it is a *well-partial-order* (wpo). Any wqo can be turned into a wpo by quotienting by the equivalence relation $\leq \cap \geq$, also denoted with $\equiv$. We say that two wqos are wpo-*isomorphic* (denoted with $\cong_{\mathsf{wpo}}$) when the wpos obtained through quotienting are isomorphic (denoted with $\cong$). We prefer working with wqos because constructions like taking subsets or multisets yield wqos even when applied to wpos.

When α is an ordinal, we may use α to denote the associated well-order $(\alpha, \in)$, which is a wqo. For $k \in \mathbb{N}$ we let Γ_k denote a wqo of k incomparable elements, i.e., a size-k antichain.

Classical operations on wqos. We will now recall the definitions of the operations on wqos that are used in this article; for a more thorough survey on these constructions and their ordinal invariants, we redirect the reader to the work of Džamonja et al. (2020). For the rest of this section, we will assume that $(A, \leq_A)$ and $(B, \leq_B)$ are two wqos.

---

[a]We use $a \perp b$ to denote that a and b are incomparable, i.e., that $a \not\leq b \wedge b \not\leq a$.

Figure 2: Cartesian product (left) and lexicographic product (right) of the wqos A_3 and Γ_2 from Example 2.2. The representation is a Hasse diagram where edges are oriented upwards.

The *disjoint sum* $(A \sqcup B, \leq_\sqcup)$ and the *lexicographic sum* $(A + B, \leq_+)$ both have for support the disjoint union of A and B. The ordering of $A \sqcup B$ is defined as $\leq_A \cup \leq_B$, whereas the ordering of $A + B$ is $\leq_A \cup \leq_B \cup \{ (a, b) \mid a \in A, b \in B \}$.

Example 2.1. *Consider the* wqos Γ_2 *and* Γ_3 *made of 2 and 3 incomparable elements, respectively. Their disjoint sum* $\Gamma_2 \sqcup \Gamma_3$ *is a* wqo *made of 5 incomparable elements, i.e.,* Γ_5. *Their lexicographic sum* $\Gamma_2 + \Gamma_3$ *is a* wqo *made of 2 levels: the bottom level contains 2 incomparable elements, and the top level contains 3 incomparable elements, each greater than elements of the bottom level.*

It is easy to see that the disjoint sum $\bigsqcup_{1 \leq i \leq n} A_i$ of finitely many wqos is a wqo, and it is immediate that an infinite disjoint sum (of non-empty) wqos is not a wqo. For the lexicographic sum, the infinite case is more interesting: given an ordinal α and an α-indexed family of wqos $(A_\gamma)_{\gamma < \alpha}$, their lexicographic sum $\sum_{\gamma < \alpha} A_\gamma$ remains a wqo.

The *Cartesian product* $(A \times B, \leq_\times)$ and the *lexicographic product* $(A \cdot B, \leq_.)$ both have for support $A \times B$, and their orderings are obtained as follows:

$$(a, b) \leq_\times (a', b') \xLeftrightarrow{\text{def}} a \leq_A a' \wedge b \leq_B b' \,,$$

$$(a, b) \leq_. (a', b') \xLeftrightarrow{\text{def}} b \leq_B b' \wedge (b \equiv_B b' \implies a \leq_A a') \,.$$

In this document, we will use $A^{\times n}$ to denote the Cartesian product of n copies of A, with n finite.

Example 2.2. *Consider the* wqos A_3 *made of 3 elements* $a_1 < a_2 < a_3$ *and* Γ_2 *made of 2 incomparable elements. Their Cartesian product* $\Gamma_2 \times A_3$ *is a* wqo *made of 6 elements: two chains* $a_1 < a_2 < a_3$ *side by side. Their lexicographic product* $\Gamma_2 \cdot A_3$ *is a* wqo *made of 3 levels: each level containing a copy of* Γ_2, *and each element of a level being greater than all elements of the lower levels. A graphical representation of these two products is given in Figure 2.*

It is known that both products are wqos when A and B are. We would like to emphasize the fact that our definition of the lexicographic product gives priority to the second component of pairs, contrary to the convention for words in a dictionary: this choice aligns lexicographic product with the usual product of ordinals.

The set of finite sequences (or *words*) over A, written $A^{<\omega}$, is ordered using the *word embedding relation* defined via:

$$\boldsymbol{u} = u_1 \cdots u_n \leq_{<\omega} \boldsymbol{v} = v_1 \cdots v_m \xLeftrightarrow{\text{def}} \exists 1 \leq f_1 < f_2 < \cdots < f_n \leq m \text{ such that } \forall 1 \leq i \leq n, u_i \leq_A v_{f_i} \,.$$

In the above definition, n and m are the lengths of $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively, and f is an embedding of $\boldsymbol{u}$ into $\boldsymbol{v}$. We use ε to denote the empty word, of length 0.

Remark 2.3. The word embedding relation can be defined inductively as follows: $\varepsilon \leq_{<\omega} \boldsymbol{v}$ for any word $\boldsymbol{v}$, and $a\boldsymbol{u} \leq_{<\omega} b\boldsymbol{v}$ iff either $a \leq_A b$ and $\boldsymbol{u} \leq_{<\omega} \boldsymbol{v}$, or $a\boldsymbol{u} \leq_{<\omega} \boldsymbol{v}$.

The *set of finite multisets* $(\mathsf{M}^{\diamond}(A), \leq_{\diamond})$ of elements in A ordered with *multiset embedding*, corresponds to $(A^{<\omega}, \leq_{<\omega})$ quotiented by the following equivalence relation: $\boldsymbol{u} \equiv \boldsymbol{v}$ iff $\boldsymbol{v}$ can be obtained from $\boldsymbol{u}$ by permuting the order of the letters. Equivalently $\boldsymbol{u} \leq_{\diamond} \boldsymbol{v}$ iff there exist some permutations $\boldsymbol{u} \equiv \boldsymbol{u}'$ and $\boldsymbol{v} \equiv \boldsymbol{v}'$ with $\boldsymbol{u}' \leq_{<\omega} \boldsymbol{v}'$ (actually, a single permutation is always sufficient).

Remark 2.4. The multiset embedding relation, as defined above, does *not* coincide with another classical ordering used on multisets due to Manna and Dershowitz (Dershowitz and Manna, 1979).

<div align="center">***</div>

Let us now introduce the main operator of this article, that is the *finitary powerset construction*. We write $\mathsf{P}_{\mathsf{f}}(A)$ for the collection of finite subsets[b] of A, with typical elements $S, S', \ldots$ and endow it with the *Hoare embedding* relation (also known as the *domination quasi-ordering*, or the *lower preorder*), defined via

$$(\text{Hoare}) \qquad\qquad S \sqsubseteq_{\mathcal{H}} S' \stackrel{\text{def}}{\iff} \forall a \in S \,\exists b \in S' \; a \leq_A b \,.$$

It is well-known that $(\mathsf{P}_{\mathsf{f}}(A), \sqsubseteq_{\mathcal{H}})$ is a wqo when A is whereas, as shown by Rado (1954), the full powerset $\mathsf{P}(A)$ (that considers infinite subsets as well) needs not be.

Remark 2.5. The literature considers other ways of ordering the powerset of ordered sets, see, e.g., (Marcone, 2001). There exist the *Smyth ordering*, or *upper preorder*, defined with $S \sqsubseteq_{\mathcal{S}} S' \stackrel{\text{def}}{\iff} \forall b \in S' \,\exists a \in S \; a \leq_A b$, and an Egli-Milner ordering obtained as the intersection of $\sqsubseteq_{\mathcal{H}}$ and $\sqsubseteq_{\mathcal{S}}$.

There is a duality between $\sqsubseteq_{\mathcal{H}}$ and $\sqsubseteq_{\mathcal{S}}$: for $S_1, S_2 \subseteq A$ one has $S_1 \sqsubseteq_{\mathcal{H}} S_2 \iff A \setminus (\downarrow S_1) \sqsubseteq_{\mathcal{S}} A \setminus (\downarrow S_2)$, where $\downarrow S$ denotes the downward-closure of a subset of A. In this article we neither consider the Smyth nor Egli-Milner orderings since they do not in general give rise to a well-quasi-ordered $\mathsf{P}_{\mathsf{f}}(A)$.

About ordinal arithmetic. To measure the complexity of wqos, we will use "ordinal invariants," and therefore need to manipulate ordinals and their arithmetic. We suppose well-known the operations of sum $\alpha + \beta$, product $\alpha \cdot \beta$, natural sum $\alpha \oplus \beta$, and natural product $\alpha \otimes \beta$ on ordinals, and refer to, e.g., Fraïssé (1986) for a more complete presentation of ordinal arithmetic. Let us point out particular ordinals that will often be special cases in our results and proofs.

Definition 2.6. An ordinal α is

- an *ε-number* whenever $\omega^{\alpha} = \alpha$;
- *additively indecomposable* when for any $\beta < \alpha$ and $\gamma < \alpha$, $\beta \oplus \gamma < \alpha$. Alternatively, additively indecomposable ordinals are of the form $\omega^{\alpha'}$ with α' any ordinal.
- *multiplicatively indecomposable* when for any $\beta < \alpha$ and $\gamma < \alpha$, $\beta \otimes \gamma < \alpha$. Alternatively, multiplicatively indecomposable ordinals (besides 2) are of the form $\omega^{\alpha'}$ with α' additively indecomposable.

As an abuse of language, we will simply write "*indecomposable*" for "multiplicatively indecomposable" in the rest of this article, and always rule out 0, 1 and 2 from our notions of indecomposable ordinals in order to simplify our statements, as these will often be treated separately.

---

[b]This collection is more usually denoted $\mathscr{P}_{\text{fin}}(A)$ in the literature, or also $\text{Fin}(A)$, $[A]^{<\omega}$, and $\mathscr{P}_{<\omega}(A)$ in some subfields.

Since some of our results rely heavily on 2-exponentiation (see Table 1), let us briefly recall that ordinal exponentiation is defined via $\alpha^0 \stackrel{\text{def}}{=} 1$, $\alpha^{\beta+1} \stackrel{\text{def}}{=} \alpha^\beta \cdot \alpha$ and, for λ a limit ordinal, $\alpha^\lambda \stackrel{\text{def}}{=} \sup_{\gamma<\lambda} \alpha^\gamma$. Exponentiation satisfies both $\alpha^{(\beta+\gamma)} = \alpha^\beta \cdot \alpha^\gamma$ and $(\alpha^\beta)^\gamma = \alpha^{(\beta\cdot\gamma)}$, which entails the following useful fact:

Fact 2.7 (2-exponentiation). *If $n < \omega$ and $\alpha = \omega \cdot \alpha' + n$ then $2^\alpha = \omega^{\alpha'} \cdot 2^n$.*

Now since any ordinal can be written (in a unique way) as $\alpha = \omega \cdot \alpha' + n$ with $n < \omega$, Fact 2.7 implies that 2^α is additively indecomposable if, and only if, α is a limit ordinal.

When constructing examples of wqos with specific ordinal invariants, we will often rely on the *Cantor Normal Form* (CNF) of ordinals. Any non-zero ordinal α can be written uniquely as a finite sum

$$\alpha = \sum_{i<n} \omega^{\alpha_i}$$

with $n > 0$ and $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{n-1}$. The ordinals α_i are the exponents of the CNF of α. For a ε-number α, its CNF has a single term, i.e., $\alpha = \omega^\alpha$.

Ordinal invariants. We follow the path of Kříž and Thomas (1990, § 4) to introduce *ordinal invariants*. For any wqo $(A, \leq_A)$, we define $\mathsf{Bad}(A)$ (respectively $\mathsf{Dec}(A)$ and $\mathsf{Inco}(A)$) as the tree of bad sequences (respectively strictly decreasing sequences and antichain sequences) of A ordered by inverse prefix order: in each tree the empty sequence is the root, and if a sequence s is a prefix of a sequence t, then $s \geq t$.

Observe that, since A is a wqo (hence satisfies the finite antichain condition and is well-founded), $\mathsf{Bad}(A)$ is a tree without infinite branches, i.e., is well-founded, and so are $\mathsf{Dec}(A)$ and $\mathsf{Inco}(A)$ since they are subtrees of $\mathsf{Bad}(A)$. However, these trees may be infinite by way of having infinite branching.

We ascribe an *ordinal rank* $\mathrm{rk}(s)$ to any node s of a well-founded tree T from bottom to top. Let $s \in T$ be a node: if s is a leaf, then $\mathrm{rk}(s)$ is defined as 0. Otherwise, $\mathrm{rk}(s)$ is defined as the supremum of the ranks of its successors plus one, i.e., $\sup\{\mathrm{rk}(t) + 1 \mid t \lessdot s \text{ in } T\}$. Since T can be infinitely branching, $\mathrm{rk}(s)$ may be an infinite ordinal. The *rank of T* is defined as the rank of its root.

Definition 2.8 (Ordinal invariants). The *maximal order type* (m.o.t.), *height* and *width* of a wqo A are respectively the rank of $\mathsf{Bad}(A)$, $\mathsf{Dec}(A)$ and $\mathsf{Inco}(A)$. They are denoted $\mathbf{o}(A)$, $\mathbf{h}(A)$ and $\mathbf{w}(A)$.

Remark 2.9. Since $\mathsf{Dec}(A)$ is well-founded when A is well-founded, the definition of height can be extended to well-founded quasi-orders.

In accordance with the inductive definition of the ordinal rank, the ordinal invariants can be computed using so-called *descent equations*. For a relation symbol $\bullet$ among, e.g., $<, \leq, \not\geq, \perp, \ldots$, and some element x of a wqo A, we define the *residual* $A_{\bullet x} = \{y \in A \mid y \bullet x\}$. Then, with the above notations the following inductive characterisations can be derived:

(Do) $$\mathbf{o}(A) = \sup_{x \in A} \left\{ \mathbf{o}(A_{\not\geq x}) + 1 \right\},$$

(Dh) $$\mathbf{h}(A) = \sup_{x \in A} \left\{ \mathbf{h}(A_{<x}) + 1 \right\},$$

(Dw) $$\mathbf{w}(A) = \sup_{x \in A} \left\{ \mathbf{w}(A_{\perp x}) + 1 \right\}.$$

The notion of residual extends to subsets $B \subseteq A$ by considering the intersection of the residuals $A_{\bullet y}$ where y ranges over B, namely:

$$A_{\bullet B} \overset{\text{def}}{=} \bigcap_{y \in B} A_{\bullet y} \,.$$

The reader may notice that $A_{<x}$ is in fact the *downward closure* of $\{x\}$ in A, a basic and useful notion that we denote with $\downarrow_A \{x\}$ or $\downarrow_A x$ as is common. However, residuation and closure are different concepts: the downward closure of some subset $B \subseteq A$, written $\downarrow_A B$, is the *union* of the closures $\downarrow_A \{y\}$ for $y \in B$, hence is *not* the same as $A_{<B}$ (unless B is a singleton). We write more simply $\downarrow x$ and $\downarrow B$ when the wqo A is clear from the context. Note that $\downarrow_A A = A$ (closures are extensive), and that $\downarrow \emptyset = \emptyset$ (they preserve the empty set). A set $B \subseteq A$ is *downwards closed* when $\downarrow_A B = B$.

Using the descent equations, we can compute our first ordinal invariants and relate them to each other. For instance, the ordinal invariants of the finite partial-order of Figure 1 are exactly its size, height and width, i.e., $\mathbf{o}(A) = 6$, $\mathbf{h}(A) = 4$ and $\mathbf{w}(A) = 3$. Similarly, it is relatively straightforward to prove that for $A = \omega$ (the natural numbers with the usual ordering), then $\mathbf{o}(A) = \mathbf{h}(A) = \omega$, and $\mathbf{w}(A) = 1$.

One important fact about the ordinal invariants that is worth mentioning is that they only count "non-equivalent" elements (strictly below, non-comparable, or not above), hence are oblivious to quotienting by equivalence. Hence, working with a wqo or with its associated wpo does not change the values of the invariants:

Fact 2.10. *Let A and B be two* wqos. *If $A \cong_{\mathsf{wpo}} B$ then $\mu(A) = \mu(B)$ for any invariant μ among* $\{\mathbf{o}, \mathbf{h}, \mathbf{w}\}$.

We summarize the current knowledge on ordinal invariants in Table 2.[c] As one can see, for most operations on wqos, the ordinal invariants can be expressed as a function of the invariants of the arguments. Actually, this is even better, one can compute any specific invariant from the same invariant of the arguments.

Table 2. : How to compute ordinal invariants compositionally (Džamonja et al., 2020).

Space	M.O.T.	Height	Width
α	α	α	1
$A \sqcup B$	$\mathbf{o}(A) \oplus \mathbf{o}(B)$	$\max(\mathbf{h}(A), \mathbf{h}(B))$	$\mathbf{w}(A) \oplus \mathbf{w}(B)$
$A + B$	$\mathbf{o}(A) + \mathbf{o}(B)$	$\mathbf{h}(A) + \mathbf{h}(B)$	$\max(\mathbf{w}(A), \mathbf{w}(B))$
$A \times B$	$\mathbf{o}(A) \otimes \mathbf{o}(B)$	$\mathbf{h}(A) \hat{\oplus} \mathbf{h}(B)$	*Not functional*
$A \cdot B$	$\mathbf{o}(A) \cdot \mathbf{o}(B)$ [d]	$\mathbf{h}(A) \cdot \mathbf{h}(B)$	$\mathbf{w}(A) \odot \mathbf{w}(B)$
$\mathsf{M}^{\diamond}(A)$	$\omega^{\widehat{\mathbf{o}(A)}}$	$h^{\star}(A)$	$\mathbf{o}(\mathsf{M}^{\diamond}(A))$ [e]
$A^{<\omega}$	$\omega^{\omega^{(\mathbf{o}(A)^{\pm})}}$	$h^{\star}(A)$	$\mathbf{o}(A^{<\omega})$ [f]
$\mathsf{P}_{\mathsf{f}}(A)$	?	?	?

---

[c]The table uses some ad-hoc ordinal operations, like $\alpha^{\pm}$ or $\alpha \hat{\oplus} \beta$, for conciseness and clarity: they all appear in earlier papers (but sometimes the notation is new) and their definition is recalled in Figure 3 (page 8).

Example 2.11. *Consider $A = (\omega + \omega) \sqcup (\omega + \omega)$. Using Table 2, we can compute its ordinal invariants compositionally as follows: $\mathbf{o}(A) = \mathbf{o}(\omega + \omega) \oplus \mathbf{o}(\omega + \omega) = \omega \cdot 4$. Similarly, $\mathbf{h}(A) = \omega \cdot 2$, and $\mathbf{w}(A) = 2$.*

Unfortunately, both the Cartesian product and the finitary powerset do not belong to this nice picture, as witnessed by the "Not functional" and "Unknown" entries in Table 2.

$$\alpha \,\hat{\oplus}\, \beta \stackrel{\text{def}}{=} \sup\{\, \alpha' \oplus \beta' \,\big|\, \alpha' < \alpha, \beta' < \beta \,\}\,,$$

$$\alpha^{\pm} \stackrel{\text{def}}{=} \begin{cases} \alpha - 1 & \text{if } \alpha \text{ is finite,} \\ \alpha + 1 & \text{if } \alpha = \varepsilon + n \text{ with } \varepsilon \text{ an } \varepsilon\text{-number and } n < \omega, \\ \alpha & \text{otherwise,} \end{cases}$$

$$h^{\star}(A) \stackrel{\text{def}}{=} \begin{cases} \mathbf{h}(A) & \text{if } \mathbf{h}(A) \text{ is additively indecomposable } \geq \omega\,, \\ \mathbf{h}(A) \cdot \omega & \text{otherwise,} \end{cases}$$

$$\alpha^{\circ} \stackrel{\text{def}}{=} \begin{cases} \alpha + 1 & \text{if } \alpha = \varepsilon + n\,, \\ \alpha & \text{otherwise.} \end{cases}$$

$$\widehat{\alpha} \stackrel{\text{def}}{=} \omega^{\alpha_1^{\circ}} + \cdots + \omega^{\alpha_n^{\circ}} \text{ when } \alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}\,,$$

$$\alpha \odot \beta \text{ defined via } \begin{cases} \alpha \odot 0 \stackrel{\text{def}}{=} 0\,, \\ \alpha \odot (\beta + 1) \stackrel{\text{def}}{=} (\alpha \odot \beta) \oplus \alpha\,, \\ \alpha \odot \lambda \stackrel{\text{def}}{=} \sup_{\gamma < \lambda}(\alpha \odot \gamma + 1) \text{ for } \lambda \text{ limit.} \end{cases}$$

Figure 3: Definition of the notations used in Table 2.

Techniques for ordinal invariants. When compositional techniques fail, one can resort to more general methods to compute or estimate the ordinal invariants of wqos. In the remainder of this section, we introduce the basic toolbox used in this article. First, since $\mathsf{Dec}(A)$ and $\mathsf{Inco}(A)$ are subtrees of $\mathsf{Bad}(A)$, we always have the following inequalities:

(1)
$$\mathbf{h}(A) \leq \mathbf{o}(A) \quad \text{and} \quad \mathbf{w}(A) \leq \mathbf{o}(A)\,.$$

In the finite case, one also knows that $\mathbf{o}(A) \leq \mathbf{h}(A) \times \mathbf{w}(A)$, where $\times$ is the usual multiplication of natural numbers. As shown by Kříž and Thomas, this generalizes to the infinite case, using the natural product, $\otimes$, on ordinals:

Theorem 2.12 (Kříž and Thomas (1990, Thm. 4.13)). *For any wqo A, $\mathbf{o}(A) \leq \mathbf{h}(A) \otimes \mathbf{w}(A)$.*

Theorem 2.12 is often used when $\mathbf{o}(A)$ is multiplicatively indecomposable in view of the following corollary:

Corollary 2.13. *Assume that $\mathbf{o}(A)$ is multiplicatively indecomposable. If $\mathbf{h}(A) < \mathbf{o}(A)$ then $\mathbf{w}(A) = \mathbf{o}(A)$, and if $\mathbf{w}(A) < \mathbf{o}(A)$ then $\mathbf{h}(A) = \mathbf{o}(A)$.*

---

[d] Assuming that $\mathbf{o}(B)$ is a limit. See **?** for the general case.
[e] Assuming that $\mathbf{o}(A) = \omega^{\alpha}$ is additively indecomposable $\geq \omega$.
[f] Assuming that $\mathbf{o}(A) > 1$.

Another useful technique to establish inequalities between ordinal invariants is to compare the invariants of different wqos. The most straightforward way to relate two wqos for us will be reflections and their special cases, substructures and augmentations.

Definition 2.14 (Augmentation). A quasi-order $(A, \leq_A)$ is an *augmentation* of $(B, \leq_B)$ whenever $A = B$ and $\leq_B \subseteq \leq_A$. We write this relation as $B \leq_{\mathsf{aug}} A$.

Definition 2.15 (Substructure). A quasi-order $(A, \leq_A)$ is a *substructure* of a quasi-order $(B, \leq_B)$ whenever $A \subseteq B$ and $\leq_A$ is the restriction $\leq_B \cap (A \times A)$ of $\leq_B$ to A. This relation is written $A \leq_{\mathsf{st}} B$.

In the remainder of this article we often write statements like, e.g., "A is a substructure of $A \times A$", when we really mean "A is *isomorphic to a* substructure of $A \times A$", or even wpo-isomorphic to a substructure of $A \times A$, since these do not influence the ordinal invariants (recall Fact 2.10).

Definition 2.16 (Reflection). A mapping $f \colon A \to B$ between two wqos is a *reflection* if $f(x) \leq_B f(y)$ implies $x \leq_A y$. We write $A \hookrightarrow B$ when there exists a reflection from A to B.

Example 2.17. *Let A be any* wqo. *The mapping $f : A \to \mathsf{P}_{\mathsf{f}}(A)$ defined by $f(x) = \{x\}$ is a reflection from A to $\mathsf{P}_{\mathsf{f}}(A)$ since $f(x) \sqsubseteq_{\mathscr{H}} f(y)$, i.e., $\{x\} \sqsubseteq_{\mathscr{H}} \{y\}$, implies $x \leq_A y$.*

In general the ordinal invariants behave monotonically with reflections and their special cases, as summarized in Lemma 2.18.

Lemma 2.18. *For any* wqos *A, B,*

- *if $A \leq_{\mathsf{st}} B$ then $\mu(A) \leq \mu(B)$ for $\mu \in \{\, \mathbf{o}, \mathbf{w}, \mathbf{h} \,\}$.*
- *if $A \geq_{\mathsf{aug}} B$ then $\mu(A) \leq \mu(B)$ for $\mu \in \{\, \mathbf{o}, \mathbf{w} \,\}$. Moreover, if A is a* wpo *then $\mathbf{h}(A) \geq \mathbf{h}(B)$.*
- *if $A \hookrightarrow B$ then $\mu(A) \leq \mu(B)$ for $\mu \in \{\, \mathbf{o}, \mathbf{w} \,\}$.*

The exception with $\mathbf{h}$ is that, when A is an augmentation of B, the pairs (a, b) in $\leq_A \setminus \leq_B$ may give rise to new descending sequences, and potentially to a larger height, but they may also give rise to new pairs of equivalent elements (if $b <_B a$ then $a \equiv_A b$) with the potential of reducing some strictly decreasing sequences from A, leading to a diminished height.

As many proofs will rely on the descent equations (Do–Dw), it will be particularly useful for us to be able to find substructures with prescribed ordinal invariants when proving lower bounds. This is always possible, as stated in Lemma 2.19.

Lemma 2.19. *For any ordinals $\beta < \alpha$, any $\mu \in \{\, \mathbf{o}, \mathbf{w}, \mathbf{h} \,\}$, and any* wqo *$A$ such that $\mu(A) = \alpha$, there exists $B \leq_{\mathsf{st}} A$ such that $\mu(B) = \beta$.*

Proof. It is well-known (see, e.g., Wolk (1967)) that when a well-founded tree has rank α then every $\beta < \alpha$ is the rank of some node of the tree. Thus if $\beta < \mathbf{o}(A)$, the tree of bad sequences has a node s (a bad sequence) of rank β. If we write S for the set of elements listed in s, the subtree rooted at s is isomorphic to $\mathsf{Bad}(A_{\not\geq S})$ so $\beta = \mathbf{o}(A_{\not\geq S})$ and we conclude by noting that the residual $A_{\not\geq S}$ is a substructure of A. Finally, the same reasoning applies to $\mathbf{h}(A)$ and $\mathbf{w}(A)$ using the trees $\mathsf{Dec}(A)$ and $\mathsf{Inco}(A)$ and residuals of the form $A_{<S}$ and $A_{\perp S}$. $\square$

Notice that $\mathsf{P}_{\mathsf{f}}(A)$ is a quotient of $\mathsf{M}^{\diamond}(A)$ through the map $M \in \mathsf{M}^{\diamond}(A) \mapsto \{\, x \in A \mid M(x) \geq 1 \,\}$. Unfortunately quotients do not preserve the ordinal invariants, and $\mathsf{M}^{\diamond}(A)$ and $\mathsf{P}_{\mathsf{f}}(A)$ actually exhibit strikingly different behaviors.

Finitary powerset. Let us finish this preliminary section by collecting few basic facts about the finitary powerset constructions that can be immediately derived from our above tools and definitions. These are folklore results, but will be used extensively in the remainder of this article.

Like all the previously recalled operations on wqos, the finitary powerset construction is monotonic with respect to the orderings associated with augmentations, substructures and reflections: if $A \leq_{\text{aug}} B$ then $P_f(A) \leq_{\text{aug}} P_f(B)$ etc.

For some proofs, it will be easier to work with wpos rather than wqos. Unfortunately, the finitary powerset construction does not preserve wpos and can introduce equivalence classes. However $P_f(A)$ is wpo-isomorphic to $\text{Ant}(A)$, the set of (finite) antichains ordered with the Hoare embedding, through the map associating with any finite $S \in P_f(A)$ the set $\max S$ of its maximal elements. If A is a wpo then $\text{Ant}(A)$ is a wpo too.

Finally, we list in Lemma 2.20 some simple isomorphisms (or wpo-isomorphisms) involving the finitary powerset construction together with disjoint unions, lexicographic sums, and ordinals.

Lemma 2.20. *Let A, B be wqos, and α be an ordinal. Then,*

(1) $P_f(A + B) \cong_{\text{wpo}} P_f(A) + (P_f(B) \setminus \{\emptyset\})$,
(2) $P_f(A \sqcup B) \cong P_f(A) \times P_f(B)$, *and*
(3) $P_f(\alpha) \cong_{\text{wpo}} 1 + \alpha$.

Proof. Any finite subset S of $A + B$ can be decomposed into a subset S_A of A and a subset S_B of B. For any two subsets S, S', if $S'_B \neq \emptyset$, then $S \sqsubseteq_{\mathcal{H}} S'$ iff $S_B \sqsubseteq_{\mathcal{H}} S'_B$. Otherwise, when $S'_B = \emptyset$, then $S \sqsubseteq_{\mathcal{H}} S'$ iff $S_B = \emptyset$ too and $S_A \sqsubseteq_{\mathcal{H}} S'_A$. Thus the function mapping S to S_B if $S_B \neq \emptyset$, and to S_A otherwise, is a monotonic reflection from $P_f(A + B)$ to $P_f(A) + (P_f(B) \setminus \{\emptyset\})$.

Similarly, any finite subset S of $A \sqcup B$ can be decomposed into a subset S_A of A and a subset S_B of B, such that $S \sqsubseteq_{\mathcal{H}} S'$ iff $S_A \sqsubseteq_{\mathcal{H}} S'_A$ and $S_B \sqsubseteq_{\mathcal{H}} S'_B$ for any finite subset S, S' of $A \sqcup B$. Thus the function mapping S to the pair (S_A, S_B) is an isomorphism between $P_f(A \sqcup B)$ and $P_f(A) \times P_f(B)$.

Finally, we observed earlier that $P_f(\alpha)$ is wpo-isomorphic to $\text{Ant}(\alpha)$. However, any non-empty finite subset of α has a unique maximum, hence, every (non-empty) antichain of α is a singleton set. When also the empty set is counted we obtain $\text{Ant}(\alpha) \cong 1 + \alpha$. $\square$

3. Upper and lower bounds for $P_f(A)$

Our first claim is that the ordinal invariants of $P_f(A)$ cannot be expressed as a function of $\mathbf{o}(A)$, $\mathbf{h}(A)$, and $\mathbf{w}(A)$, as witnessed in Example 3.1.

Example 3.1. *Consider $A_1 = (\omega + \omega) \sqcup (\omega + \omega)$ and $A_2 = (\omega \sqcup \omega) + (\omega \sqcup \omega)$. These two wqos have the same ordinal invariants, but $P_f(A_1)$ and $P_f(A_2)$ disagree on all 3 ordinal invariants.*

Proof of Claim. Observe that $P_f(A_1) \cong \omega \cdot 2 \times \omega \cdot 2$, and that $P_f(A_2) \cong_{\text{wpo}} (\omega \times \omega) + (\omega \times \omega)$ (Lemma 2.20). As a consequence, we obtain the following ordinal invariants[g]:

---

[g]In these and later computations, we rely on Abraham (1987) and Vialard (2024b) for the width of Cartesian products of ordinals.

A	o	h	w
A_1	$\omega \cdot 4$	$\omega \cdot 2$	2
A_2	$\omega \cdot 4$	$\omega \cdot 2$	2
$\mathsf{P_f}(A_1)$	$\omega^2 \cdot 4$	$\omega \cdot 3$	$\omega \cdot 3$
$\mathsf{P_f}(A_2)$	$\omega^2 \cdot 2$	$\omega \cdot 2$	ω

$\square$

The non-functionality of the ordinal invariants of $\mathsf{P_f}(A)$ does not prevent us from building bounds for **h**, **w**, and **o**. Our main argument to prove upper and lower bounds will be structural in the sense that, from the form of the ordinal invariants of A (e.g., are they successor or limit ordinals? finite or infinite? indecomposable?), we will deduce some structure B related to A such that the invariants of $\mathsf{P_f}(B)$ are easier to estimate. Here is a first example, a structural lemma used for the upper bound on the maximal order type:

Lemma 3.2 (Sandwich Lemma). *Let $(A, \leq)$ be a* wqo *such that $\mathbf{o}(A) = \alpha + \beta$ for some ordinals α, β. Then there exists a partition $A = A_\alpha \uplus A_\beta$ of A with $\mathbf{o}(A_\alpha) = \alpha$, $\mathbf{o}(A_\beta) = \beta$, and such that $A_\alpha \sqcup A_\beta \leq_{\mathsf{aug}} A \leq_{\mathsf{aug}} A_\alpha + A_\beta$.*

Section A.1 contains the proof of Lemma 3.2, which is a refinement of a previous result from (de Jongh and Parikh, 1977, Theorem 3.2), where the case $\beta = 1$ is handled.

3.1 Maximal Order Type of $\mathsf{P_f}(A)$

Let A be a wqo. It is clear that, modulo isomorphism, $1 + A$ is a substructure of $\mathsf{P_f}(A)$ through the map $0 \mapsto \emptyset$ and, for $x \in A$, $x \mapsto \{ x \}$. Using Table 2 and Lemma 2.18, we obtain the following lower bound:

$$(2) \qquad\qquad 1 + \mathbf{o}(A) = \mathbf{o}(1 + A) \leq \mathbf{o}(\mathsf{P_f}(A)) \,.$$

For the upper bound, we will rely on Lemma 3.2 to recursively decompose A and prove that $\mathbf{o}(\mathsf{P_f}(A)) \leq 2^{\mathbf{o}(A)}$.

Theorem 3.3. *For all* wqo *A, $1 + \mathbf{o}(A) \leq \mathbf{o}(\mathsf{P_f}(A)) \leq 2^{\mathbf{o}(A)}$.*

Proof. We prove the second inequality by induction on $\mathbf{o}(A)$. We consider three cases:

1. $\mathbf{o}(A)$ is finite: Assume w.l.o.g. that A is a wpo. Then $\mathbf{o}(A)$ is the cardinal of A and $\mathbf{o}(\mathsf{P_f}(A))$ is less than, or equal to, the cardinal of $\mathsf{P_f}(A)$ which is $2^{\mathbf{o}(A)}$.
2. $\mathbf{o}(A)$ is an infinite successor ordinal: Then $\mathbf{o}(A) = \alpha + n$ with α limit and infinite, and $1 \leq n < \omega$. Using Lemma 3.2 we can split A as $A = A_1 \uplus A_2$ with $\mathbf{o}(A_1) = \alpha$, $\mathbf{o}(A_2) = n$, and $A \geq_{\mathsf{aug}} A_1 \sqcup A_2$. Because $\mathsf{P_f}$ is monotonic with respect to augmentations, we see that $\mathsf{P_f}(A) \geq_{\mathsf{aug}} \mathsf{P_f}(A_1 \sqcup A_2)$. Then, from Lemma 2.18, we deduce that $\mathbf{o}(\mathsf{P_f}(A)) \leq \mathbf{o}(\mathsf{P_f}(A_1 \sqcup A_2))$. But we know from Lemma 2.20 that $\mathsf{P_f}(A_1 \sqcup A_2) \cong_{\mathsf{wpo}} \mathsf{P_f}(A_1) \times \mathsf{P_f}(A_2)$, and from Table 2 that the Cartesian product is the commutative product $\otimes$ of the two maximal order types, hence we obtain:

$$\mathbf{o}(\mathsf{P_f}(A)) \leq \mathbf{o}(\mathsf{P_f}(A_1) \times \mathsf{P_f}(A_2)) = \mathbf{o}(\mathsf{P_f}(A_1)) \otimes \mathbf{o}(\mathsf{P_f}(A_2)) \,.$$

The induction hypothesis gives $\mathbf{o}(\mathsf{P_f}(A_1)) \leq 2^\alpha$ and $\mathbf{o}(\mathsf{P_f}(A_2)) \leq 2^n$. Therefore, $\mathbf{o}(\mathsf{P_f}(A)) \leq 2^\alpha \otimes 2^n = 2^{\alpha+n}$. We have proven that $\mathbf{o}(\mathsf{P_f}(A)) \leq 2^{\mathbf{o}(A)}$ in this case.

3. $\mathbf{o}(A)$ is a limit ordinal: We use the descent equations:

$$(3) \qquad \mathbf{o}(\mathsf{P_f}(A)) \overset{\text{(Do)}}{=} \sup_{S \in \mathsf{P_f}(A)} \left(\mathbf{o}(\mathsf{P_f}(A)_{\not\sqsupseteq_{\mathscr{H}} S}) + 1 \right).$$

Given a finite set $S \in \mathsf{P_f}(A)$, we further decompose the set $\mathsf{P_f}(A)_{\not\sqsupseteq_{\mathscr{H}} S}$:

$$\begin{aligned}
\mathsf{P_f}(A)_{\not\sqsupseteq_{\mathscr{H}} S} &= \{T \in \mathsf{P_f}(A) \mid S \not\sqsubseteq_{\mathscr{H}} T\} \\
&= \{T \in \mathsf{P_f}(A) \mid \exists x \in S, \forall y \in T, x \not\leq y\} \\
&= \{T \in \mathsf{P_f}(A) \mid \exists x \in S, T \in \mathsf{P_f}(A_{\not\geq x})\} \\
&= \bigcup_{x \in S} \mathsf{P_f}(A_{\not\geq x}).
\end{aligned}$$

As a consequence, $\mathsf{P_f}(A)_{\not\sqsupseteq_{\mathscr{H}} S}$ is an augmentation of the disjoint sum $\bigsqcup_{x \in S} \mathsf{P_f}(A_{\not\geq x})$, and the following inequality holds with reasoning similar to the successor case (using Lemmas 2.18 and 2.20 and Table 2):

$$(4) \qquad \mathbf{o}(\mathsf{P_f}(A)_{\not\sqsupseteq_{\mathscr{H}} S}) \leq \bigoplus_{x \in S} \mathbf{o}(\mathsf{P_f}(A_{\not\geq x})) \leq \left[\max_{x \in S} \mathbf{o}(\mathsf{P_f}(A_{\not\geq x})) \right] \otimes |S|.$$

Let us write $\beta = \max_{x \in S} \mathbf{o}(A_{\not\geq x})$. Eq. (Do) entails $\beta < \mathbf{o}(A)$. By induction hypothesis, for all $x \in S$, we have $\mathbf{o}(\mathsf{P_f}(A_{\not\geq x})) \leq 2^{\mathbf{o}(A_{\not\geq x})} \leq 2^\beta$. Hence

$$\mathbf{o}(\mathsf{P_f}(A)_{\not\sqsupseteq_{\mathscr{H}} S}) \leq 2^\beta \otimes |S| < 2^{\mathbf{o}(A)},$$

where the last step relies on $2^{\mathbf{o}(A)}$ being additively indecomposable when α is limit. Using again this fact that $2^{\mathbf{o}(A)}$ is additively indecomposable, we see that $\mathbf{o}(\mathsf{P_f}(A)_{\not\sqsupseteq_{\mathscr{H}} S}) + 1 < 2^{\mathbf{o}(A)}$. Since this holds for any S, Eq. (3) entails $\mathbf{o}(\mathsf{P_f}(A)) \leq \mathbf{o}(A)$. $\qquad \square$

Notice that in the proof of Theorem 3.3, we relate the residual of a set S in $\mathsf{P_f}(A)$ with the residuals $A_{\not\geq x}$ of the elements of S in A. This method will be used several times in this article.

3.2 Height of $\mathsf{P_f}(A)$

Let A be a wqo. For the same reason as for Eq. (2), that is, because $1 + A$ is a substructure of $\mathsf{P_f}(A)$, we have the following lower bound:

$$(5) \qquad\qquad\qquad 1 + \mathbf{h}(A) \leq \mathbf{h}(\mathsf{P_f}(A)).$$

Before going further, we would like to produce a non-tight upper bound for $\mathbf{h}(\mathsf{P_f}(A))$, that relates it to the height of $\mathsf{P}(A)$ (recall that the height is defined whenever the order is well-founded). While crude, this bound will serve later in **??**.

Remark 3.4. $\mathsf{P_f}(A)$ is a substructure of the full powerset $\mathsf{P}(A)$ ordered with Hoare's embedding, which is wpo-isomorphic to $(\text{Down}(A), \subseteq)$, the downward-closed subsets of A ordered by inclusion. Therefore, $\mathbf{h}(\mathsf{P_f}(A)) \leq \mathbf{h}(\text{Down}(A)) = \mathbf{o}(A) + 1$ (Džamonja et al., 2020, Theorem 3.5).

One can improve this upper bound. As in the proof of Theorem 3.3, we rely on a structural decomposition lemma to handle the successor case, and on the relationship between the residuals of $\mathsf{P_f}(A)$ and those of A for the limit case. Since the structural lemma is simpler to obtain, we include its proof.

Lemma 3.5. *Let A be a* wqo *such that* $\mathbf{h}(A) = \alpha + 1$. *Then there exists a partition* $A = A_\perp \uplus$ $A_\top \leq_{\mathsf{aug}} A_\perp + A_\top$ *such that* $\mathbf{h}(A_\perp) = \alpha$ *and* $A_\top$ *is* wpo-*isomorphic to an antichain.*

Proof. Without any loss of generality, let us assume that A is a wpo. In this proof we use the same result of Wolk (1967) that was used in Lemma 2.19. The rank function rk of $\mathsf{Dec}(A)$ can be used to associate a rank with every element of A as follows: the rank of x is $\mathsf{rk}((x))$, referring to (x), the decreasing sequence of length 1 built with x.

Observe that elements of equal rank always forms an antichain, and that the direct sum of the antichains obtained from the partition of A by rank is an augmentation of A

If we now let $A_\top = \{ x \in A \mid \mathsf{rk}((x)) = \alpha \}$ and $A_\perp = A \setminus A_\top$, then $\mathbf{h}(A_\perp) = \alpha$ and $A_\top$ is an antichain, which concludes the proof. $\qquad\square$

Theorem 3.6. *For all* wqo *A, there exists $m \in \mathbb{N}$ such that,*

$$1 + \mathbf{h}(A) \leq \mathbf{h}(\mathsf{P_f}(A)) \leq \begin{cases} 2^{\mathbf{h}(A)} & \text{when } \mathbf{h}(A) \text{ is a limit ordinal,} \\ 2^{\mathbf{h}(A)} \cdot m & \text{when } \mathbf{h}(A) \text{ is a successor ordinal.} \end{cases}$$

Proof. There only remains to prove the upper bound. We do this by induction on $\mathbf{h}(A)$.

If $\mathbf{h}(A)$ is finite, then because A is a wqo, it is wpo-isomorphic to a finite set, and so $\mathsf{P_f}(A)$ is too. As a consequence, $\mathbf{h}(\mathsf{P_f}(A)) \leq m$ for some $m \in \mathbb{N}$.

If $\mathbf{h}(A) = \alpha + 1$ is an infinite successor ordinal, we rely on Lemma 3.5 and pick a decomposition $A = A_\perp \uplus A_\top$ such that $\mathbf{h}(A_\perp) = \alpha$ and $A_\top \cong_{\mathsf{wpo}} \Gamma_m$ for some $1 \leq m < \omega$. Without loss of generality, assume that A is a wpo. Recall that $\mathsf{P_f}(A) \cong_{\mathsf{wpo}} \mathsf{Ant}(A)$, the wpo of antichains of A ordered by Hoare's embedding and observe that any antichain S of A can be decomposed as an antichain of $S_\perp$ of $A_\perp$ and an antichain $S_\top$ of $A_\top$. Let S, S' be two antichains of A. Then $S \sqsubseteq_{\mathscr{H}} S'$ only if either $S_\top \sqsubset_{\mathscr{H}} S'_\top$, or $S_\top = S'_\top$ and $S_\top \sqsubseteq_{\mathscr{H}} S'_\top$. Hence, antichains of A can be seen as lexicographically ordered pairs of antichains of $A_\top$ and antichains of $A_\perp$:

$$\mathsf{Ant}(A) \leq_{\mathsf{aug}} \{ (S_\perp, S_\top) \in \mathsf{Ant}(A_\perp) \cdot \mathsf{Ant}(A_\top) \mid S_\perp \cup S_\top \in \mathsf{Ant}(A) \}$$
$$\leq_{\mathsf{st}} \mathsf{Ant}(A_\perp) \cdot \mathsf{Ant}(A_\top) .$$

Because $\mathbf{h}$ is monotonic with respect to augmentations and substructures for wpos (Lemma 2.18), we conclude that $\mathbf{h}(\mathsf{Ant}(A)) \leq \mathbf{h}(\mathsf{Ant}(A_\perp)) \cdot \mathbf{h}(\mathsf{Ant}(A_\top))$. From Table 2, we then conclude that $\mathbf{h}(\mathsf{Ant}(A_\perp) \cdot \mathsf{Ant}(A_\top)) = \mathbf{h}(\mathsf{Ant}(A_\perp)) \cdot \mathbf{h}(\mathsf{Ant}(A_\top))$. From the induction hypothesis, there exists $m \in \mathbb{N}$ such that $\mathbf{h}(\mathsf{Ant}(A_\perp)) \leq 2^\alpha \cdot m$ for some m, and $\mathbf{h}(\mathsf{Ant}(A_\top)) = \mathbf{h}(\Gamma_{m'})$ for some m'. Therefore, again using Table 2, $\mathbf{h}(\mathsf{Ant}(A)) \leq 2^\alpha \cdot m \cdot \mathbf{h}(\Gamma_{m'})$. We conclude, as desired, that $\mathbf{h}(\mathsf{P_f}(A)) \leq 2^\alpha \cdot (m \times (1 + m'))$.

If $\mathbf{h}(A) = \alpha$ where α is limit ordinal, then using Eq. (Dh):

$$\mathbf{h}(\mathsf{P_f}(A)) = \sup_{S \in \mathsf{P_f}(A)} \left(\mathbf{h}(\mathsf{P_f}(A)_{\sqsubset_{\mathscr{H}} S}) + 1 \right) .$$

Let us fix $S \in \mathsf{P_f}(A)$ and let $B \stackrel{\mathrm{def}}{=} \mathsf{P_f}(A)_{\sqsubset_{\mathscr{H}} S}$. Then

$$B = \{ T \in \mathsf{P_f}(A) \mid T \sqsubset_{\mathscr{H}} S \}$$
$$\subseteq \{ T \in \mathsf{P_f}(A) \mid T \subseteq {\downarrow} S \}$$
$$= \mathsf{P_f}({\downarrow} S) .$$

We recall that ${\downarrow} S \stackrel{\mathrm{def}}{=} \bigcup_{x \in S} A_{\leq x}$. Because $\mathbf{h}(A) = \alpha$ is a limit ordinal, for all $x \in A$, $\mathbf{h}(A)_{\leq x} = \mathbf{h}(A_{<x}) + 1 < \alpha$. Since $S \subseteq A$ is finite, $\mathbf{h}({\downarrow} S) < \alpha$.

We can therefore apply the induction hypothesis: There exists $m \in \mathbb{N}$ such that $\mathbf{h}(B) \leq 2^{\mathbf{h}(\downarrow S)} < 2^{\alpha}$ if $\mathbf{h}(B)$ is limit, and $\mathbf{h}(B) \leq 2^{\mathbf{h}(\downarrow S)} \cdot m < 2^{\alpha}$ otherwise.

This proves that $\mathbf{h}(\mathsf{P_f}(A)) \leq 2^{\alpha}$. $\square$

Note that the upper bound of Theorem 3.6 could be refined further to control the constant m in the successor case. To that end, one could change the induction so that in the successor case, one decomposes $\mathbf{h}(A)$ as $\alpha + n$ with α limit, and then see that m is bounded by $\mathbf{h}(\mathsf{P_f}(\Gamma_n))$. As this complexifies both the proof and the statement with low added value, we refrained from doing so.

3.3 Width of $\mathsf{P_f}(A)$

Thanks to Theorem 3.3, and since $\mathbf{w} \leq \mathbf{o}$, we see that $\mathbf{w}(\mathsf{P_f}(A)) \leq 2^{\mathbf{o}(A)}$. This inequality does not bound the width of $\mathsf{P_f}(A)$ by an expression depending on $\mathbf{w}(A)$, and we now claim that there exists no such upper bound, anticipating the tightness results of Section 4.

Example 3.7. *For all infinite ordinal α, let $A_\alpha = \omega \sqcup \alpha$. Then $\mathbf{w}(A_\alpha) = 2$ and $\mathbf{w}(\mathsf{P_f}(A_\alpha)) = \alpha$.*

Proof. Lemma 2.20 gives $\mathsf{P_f}(A_\alpha) \cong \mathsf{P_f}(\omega) \times \mathsf{P_f}(\alpha) \cong \omega \times \alpha$. There remains to compute the width of a Cartesian product of ordinals: $\mathbf{w}(\omega \times \alpha) = \alpha$ (Abraham, 1987; Vialard, 2024*b*). $\square$

As a consequence, we will focus on providing a lower bound for the width of $\mathsf{P_f}(A)$. As opposed to the case of $\mathbf{h}$ and $\mathbf{o}$, this lower bound happens to be non-trivial to obtain. Our first goal is to provide structural lemmas analogue to Lemmas 3.2 and 3.5.

Lemma 3.8. *Let A be a* wqo *such that $\mathbf{w}(A) = \alpha + n$ with $0 \leq n < \omega$. Then there exists B such that $B \sqcup \Gamma_n \leq_{\mathsf{st}} A$ and $\mathbf{w}(B) = \alpha$.*

Proof. By induction on n. We build an antichain $X = \{x_1, \ldots, x_n\}$ of A such that $\mathbf{w}(A_{\perp X}) = \alpha$ so letting $B = A_{\perp X}$ proves the claim. The case $n = 0$ is trivial.

When $n > 0$, and according to Eq. (Dw), $\mathbf{w}(A) = \sup_{x \in A}(\mathbf{w}(A_{\perp x}) + 1)$, therefore there exists $x \in A$ such that $\mathbf{w}(A_{\perp x}) = \alpha + (n - 1)$. By ind. hyp., there is an antichain $X' = \{x_1, \ldots, x_{n-1}\}$ of $A_{\perp x}$ such that $\mathbf{w}(A_{\perp x \cup X'}) = \alpha$. Now, and since X' is in $A_{\perp x}$, letting $X = \{x\} \cup X'$ yields an antichain and concludes the proof. $\square$

Corollary 3.9. *Let A be a* wqo. *If $\mathbf{w}(A) = k < \omega$, then $\Gamma_k \leq_{\mathsf{st}} A$.*

A lower bound using the structural decomposition lemmas above will naturally rely on computing widths of the form $\mathbf{w}(\mathsf{P_f}(B \sqcup \Gamma_k))$, which can be rewritten as $\mathbf{w}(\mathsf{P_f}(B) \times \mathsf{P_f}(\Gamma_k))$ according to Lemma 2.20. One major roadblock to such a proof scheme is that Table 2 does not provide a closed expression for the width of a cartesian product. To tackle this issue, we leverage the work of Džamonja et al. (2020) and Vialard (2024*b*).

Lemma 3.10 (Lower bound on the width of the Cartesian product). *Let A, B be two* wqos *such that $\mathbf{w}(B) \geq \omega$ is additively indecomposable. Then $\mathbf{w}(A \times B) \geq \mathbf{w}(B) \cdot \mathbf{o}(A)$.*

Since the proof of Lemma 3.10 uses technical notions that are not central to this article, we refer the reader to Section A.2 for details.

In the following bounds for the width of the finitary powerset, we abstract away the finite constants (as in Theorem 3.6) to simplify the analysis. Note that the finite parts are computed as $\mathbf{w}(\mathsf{P_f}(\Gamma_k))$ for some k, which is $\binom{k}{\lfloor k/2 \rfloor}$ by Sperner's theorem.

Theorem 3.11. *Let A be a* wqo. *If* $\mathbf{w}(A)$ *is infinite then* $\mathbf{w}(\mathsf{P_f}(A)) \geq 2^{\mathbf{w}(A)}$. *If* $\mathbf{w}(A) = k < \omega$ *then* $\mathbf{w}(\mathsf{P_f}(A)) \geq \mathbf{w}(\mathsf{P_f}(\Gamma_k))$.

Proof. We prove the result by induction on $\mathbf{w}(A)$.

If $\mathbf{w}(A) = k < \omega$ then by Corollary 3.9, $\Gamma_k \leq_{\mathrm{st}} A$, hence by monotonicity of $\mathsf{P_f}$, $\mathsf{P_f}(\Gamma_k) \leq_{\mathrm{st}} \mathsf{P_f}(A)$. Since the ordinal width is monotone with respect to substructures (Lemma 2.18), we conclude that

$$\mathbf{w}(\mathsf{P_f}(\Gamma_k)) \leq \mathbf{w}(\mathsf{P_f}(A)) .$$

If $\mathbf{w}(A) = \alpha + n$ with α limit and $1 \leq n < \omega$ then by Lemma 3.8, there exists a wqo B of width α, such that $B \sqcup \Gamma_n \leq_{\mathrm{st}} A$. Hence $\mathbf{w}(\mathsf{P_f}(A)) \geq \mathbf{w}(\mathsf{P_f}(B \sqcup \Gamma_n)) = \mathbf{w}(\mathsf{P_f}(B) \times \mathsf{P_f}(\Gamma_n))$ according to Lemmas 2.18 and 2.20. By induction hypothesis, $2^\alpha \leq \mathbf{w}(\mathsf{P_f}(B))$. Therefore, by Lemma 2.19, there exists $C \leq_{\mathrm{st}} \mathsf{P_f}(B)$ such that $\mathbf{w}(C) = 2^\alpha$. The latter is an additively indecomposable ordinal since α is limit and infinite, hence we may continue with

$$\mathbf{w}(\mathsf{P_f}(A)) \geq \mathbf{w}(\mathsf{P_f}(B) \times \mathsf{P_f}(\Gamma_n)) \geq \mathbf{w}(C \times \mathsf{P_f}(\Gamma_n)),$$

and with Lemma 3.10

$$\geq \mathbf{w}(C) \cdot \mathbf{o}(\mathsf{P_f}(\Gamma_n)) = 2^\alpha \cdot 2^n = 2^{\alpha+n},$$

which is the expected lower bound.

If $\mathbf{w}(A) = \omega$ then for all $k < \omega$, $\mathbf{w}(\mathsf{P_f}(A)) \geq \mathbf{w}(\mathsf{P_f}(\Gamma_k))$. Observe that $\sup_{k<\omega} \mathbf{w}(\mathsf{P_f}(\Gamma_k)) = \omega$, and therefore $\mathbf{w}(\mathsf{P_f}(A)) \geq \omega = 2^\omega$.

If $\mathbf{w}(A) = \alpha > \omega$ and α is a limit ordinal, then for all $\omega < \beta < \alpha$, there exists (Lemma 2.19) a $B_\beta \leq_{\mathrm{st}} A$ such that $\mathbf{w}(B_\beta) = \beta$. By induction hypothesis, and since $\omega < \beta$, $2^\beta \leq \mathbf{w}(\mathsf{P_f}(B)) \leq \mathbf{w}(\mathsf{P_f}(A))$. Therefore,

$$\mathbf{w}(\mathsf{P_f}(A)) \geq \sup_{\beta<\alpha} 2^\beta = 2^\alpha . \qquad \square$$

4. Tightness of the Bounds

The upper and lower bounds provided in Section 3 allow huge gaps. For instance, when $\mathbf{h}(A) = \omega^2$, Theorem 3.6 shows that $\mathbf{h}(\mathsf{P_f}(A))$ can lie between $1 + \omega^2$, i.e., ω^2, and 2^{ω^2}, i.e., ω^ω. We proceed to prove that these gaps cannot be avoided by exhibiting, for every measure $\mu \in \{\, \mathbf{w}, \mathbf{o}, \mathbf{h} \,\}$, for upper and lower bounds, a family $(X_\alpha)_\alpha$ of wqos such that $\mu(X_\alpha) = \alpha$ and such that $\mu(\mathsf{P_f}(X_\alpha))$ equals the provided (upper or lower) bound.

4.1 Tight lower bound for $\mathbf{w}(\mathsf{P_f}(A))$ and upper bound for $\mathbf{o}(\mathsf{P_f}(A))$

We first introduce a family of wqos $(\mathscr{H}_\alpha)_\alpha$ that will reach both the lower bounds for $\mathbf{w}(\mathsf{P_f}(X))$ and the upper bound for $\mathbf{o}(\mathsf{P_f}(X))$. We recall that, for any $k < \omega$, Γ_k denotes a wqo of k incomparable elements. The definition of $\mathscr{H}_\alpha$ will use the Cantor normal form (CNF) of α, i.e., the unique way of writing α as $\alpha = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_n}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$.

Definition 4.1. We first define inductively the $\mathscr{H}_\alpha$ for α a power of ω:
- $\mathscr{H}_{\omega^0} \overset{\text{def}}{=} \Gamma_1$,
- $\mathscr{H}_{\omega^1} \overset{\text{def}}{=} \sum_{n \in \omega} \Gamma_n = \Gamma_0 + \Gamma_1 + \Gamma_2 + \cdots$, see Figure 4,
- for $\alpha > 0$, $\mathscr{H}_{\omega^{\omega^\alpha}} = \sum_{\beta < \omega^\alpha} \mathscr{H}_{\omega^\beta}$,

473   –   and   for $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ in   Cantor   normal   form,   with $n > 1$, $\mathcal{H}_{\omega^\alpha} \stackrel{\text{def}}{=}$
474   $\mathcal{H}_{\omega^{\omega^{\alpha_1}}} \cdot \mathcal{H}_{\omega^{\omega^{\alpha_2}}} \cdots \cdot \mathcal{H}_{\omega^{\omega^{\alpha_n}}}$ (the lexicographic product of the family of $\mathcal{H}_{\omega^{\omega^{\alpha_i}}}$).
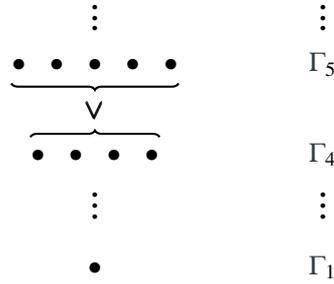

Figure 4: $\mathcal{H}_{\omega^1} \stackrel{\text{def}}{=} \Gamma_0 + \Gamma_1 + \Gamma_2 + \cdots$

475   **Proposition 4.2.** *For all ordinal α:*

$$\mathbf{h}(\mathcal{H}_{\omega^\alpha}) = \mathbf{w}(\mathcal{H}_{\omega^\alpha}) = \mathbf{o}(\mathcal{H}_{\omega^\alpha}) = \omega^\alpha.$$

476   The proof of Proposition 4.2 is done by induction on α and follows from Table 2, except that
477   some technical care is needed to handle the lexicographic product case. Therefore, the full proof
478   can be found in Section B.1.
479   Unfortunately, the family $\mathcal{H}_{\omega^\alpha}$ only covers ordinals of the form ω^α. We now extend it to
480   all ordinals as follows: for an ordinal α in Cantor normal form $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, we let $\mathcal{H}_\alpha$ be
481   defined as $\mathcal{H}_{\omega^{\alpha_1}} \sqcup \cdots \sqcup \mathcal{H}_{\omega^{\alpha_n}}$. While this ensures $\mathbf{w}(\mathcal{H}_\alpha) = \mathbf{o}(\mathcal{H}_\alpha) = \alpha$ for all α, observe that
482   $\mathcal{H}_\alpha$ has height $\max_i \omega^{\alpha_i}$ and not α.
483   There remains to evaluate the ordinal invariants of $\mathsf{P}_f(\mathcal{H}_\alpha)$.

484   **Theorem 4.3.** *For every α, $\mathbf{o}(\mathcal{H}_\alpha) = \mathbf{w}(\mathcal{H}_\alpha) = \alpha$. Furthermore, $\mathbf{w}(\mathsf{P}_f(\mathcal{H}_\alpha))$ attains the lower*
485   *bound of Theorem 3.11 and $\mathbf{o}(\mathsf{P}_f(\mathcal{H}_\alpha))$ attains the upper bound of Theorem 3.3.*

486   *Proof.* From Theorems 3.3 and 3.11 we conclude that for α infinite:

$$2^\alpha = 2^{\mathbf{w}(\mathcal{H}_\alpha)} \leq \mathbf{w}(\mathsf{P}_f(\mathcal{H}_\alpha)) \leq \mathbf{o}(\mathsf{P}_f(\mathcal{H}_\alpha)) \leq 2^{\mathbf{o}(\mathcal{H}_\alpha)} = 2^\alpha .$$

487   It follows that $\mathbf{w}(\mathsf{P}_f(\mathcal{H}_\alpha)) = \mathbf{o}(\mathsf{P}_f(\mathcal{H}_\alpha)) = 2^\alpha$. Hence, the upper bound in Theorem 3.3 and the
488   lower bound in Theorem 3.11 are attained.
489   Lastly, for $\alpha = k$ finite, $\mathcal{H}_k = \Gamma_1 \sqcup \cdots \sqcup \Gamma_1 \cong \Gamma_k$. Now $\mathbf{w}(\mathsf{P}_f(\Gamma_k))$ is precisely the lower bound
490   given in Theorem 3.11, while $(\mathsf{P}_f(\Gamma_k), \sqsubseteq_{\mathcal{H}}) = (\mathsf{P}_f(\Gamma_k), \subseteq)$ is a wpo of cardinal 2^k, hence of
491   maximal order type 2^k too.   □

492   **4.2 Tight upper bounds for $\mathbf{h}(\mathsf{P}_f(A))$**
493   We now provide a family of wqos reaching the upper bound for the height, that is: $\mathbf{h}(\mathscr{P}_\alpha) = \alpha$,
494   and $\mathbf{h}(\mathsf{P}_f(\mathscr{P}_\alpha)) = 2^\alpha$. For this recall that indecomposable ordinals are of the form ω^{ω^β}.

Definition 4.4. Define $\mathscr{P}_\alpha$ as follows:

— $\mathscr{P}_\omega \overset{\text{def}}{=} \omega$,

— if $\alpha \geq \omega^\omega$ is indecomposable, $\mathscr{P}_\alpha \overset{\text{def}}{=} \mathsf{P}_{\mathsf{f}}(\alpha^{<\omega})$.

— if $\alpha = \beta + m$ with β indecomposable and $m < \omega$, $\mathscr{P}_{\alpha+1,m} \overset{\text{def}}{=} (\mathscr{P}_\alpha + 1) \times \Gamma_m$.

Since $2^\omega = \omega$ and $\mathsf{P}_{\mathsf{f}}(\mathscr{P}_\omega) \cong_{\mathsf{wpo}} \omega$, we see that $\mathscr{P}_\omega$ reaches the upper bound for the ordinal height of the finitary powerset. For any indecomposable $\alpha \geq \omega^\omega$, $\mathscr{P}_\alpha$ is an elementary wqo (as defined in the next section). Therefore, we will see in Section 5.2 that $\mathbf{h}(\mathscr{P}_\alpha) = \alpha$, and $\mathbf{h}(\mathsf{P}_{\mathsf{f}}(\mathscr{P}_\alpha)) = 2^\alpha$. We extend the family $\mathscr{P}_\alpha$ to successors of indecomposable ordinals as follows.

Definition 4.5 (Extended Family). Let α be an indecomposable ordinal and $m < \omega$. Then,
$$\mathscr{P}_{\alpha+1,m} \overset{\text{def}}{=} (\mathscr{P}_\alpha + 1) \times \Gamma_m.$$

Lemma 4.6. *Let α be an indecomposable ordinal and $m < \omega$. Then,*
$$\mathbf{h}(\mathscr{P}_{\alpha+1,m}) = \alpha + 1 \text{ and } \mathbf{h}(\mathsf{P}_{\mathsf{f}}(\mathscr{P}_{\alpha+1,m})) \geq 2^\alpha \cdot m\,.$$

Proof. Observe that $(\mathscr{P}_\alpha + 1) \times \Gamma_m$ is isomorphic to $(\mathscr{P}_\alpha + 1) \sqcup \cdots \sqcup (\mathscr{P}_\alpha + 1)$, the disjoint sum of m copies of $(\mathscr{P}_\alpha + 1)$. Therefore $\mathbf{h}(\mathscr{P}_{\alpha+1,m}) = \mathbf{h}(\mathscr{P}_\alpha + 1) = \alpha + 1$ according to Table 2. According to Lemma 2.20, $\mathsf{P}_{\mathsf{f}}(\mathscr{P}_{\alpha+1,m}) \cong (\mathsf{P}_{\mathsf{f}}(\mathscr{P}_\alpha + 1))^{\times m} \cong (\mathsf{P}_{\mathsf{f}}(\mathscr{P}_\alpha) + 1)^{\times m}$ hence $\mathbf{h}(\mathsf{P}_{\mathsf{f}}(\mathscr{P}_{\alpha+1,m})) = \mathbf{h}(\mathsf{P}_{\mathsf{f}}(\mathscr{P}_\alpha) + 1) \hat{\oplus} \cdots \hat{\oplus} \mathbf{h}(\mathsf{P}_{\mathsf{f}}(\mathscr{P}_\alpha) + 1) \geq (2^\alpha + 1) \hat{\oplus} \cdots \hat{\oplus} (2^\alpha + 1) = 2^\alpha \cdot m + 1$. $\qquad\square$

5. An algebra of well-behaved wqos

The results in Sections 3 and 4 show that one cannot derive the height, width and maximal order type of $\mathsf{P}_{\mathsf{f}}(A)$ from the invariants of A. Hence the powerset construction behaves differently from other operations for which ordinal invariants can be computed in a compositional way (see Table 2).

In this section we show that the situation is not so negative. We consider a rather large family of "elementary" wqos closed under most classical operations, for which we can compute the ordinal invariants of $\mathsf{P}_{\mathsf{f}}(A)$, even in the presence of nested powerset constructions. This extends previous results in the literature for families of wqos which do not include the finitary powerset construction (Vialard, 2024*b*). Most notably, the family of elementary wqos *excludes* the lexicographic sum, that behaves badly with respect to the Cartesian product as noticed by Vialard (2024*b*).

Definition 5.1 (Elementary wqo). The family of *elementary* wqos is given by the following abstract grammar
$$E \overset{\text{def}}{=} \alpha \geq \omega^\omega \text{ indecomposable} \mid E_1 \sqcup E_2 \mid E_1 \times E_2 \mid E^{<\omega} \mid \mathsf{M}^\diamond(E) \mid \mathsf{P}_{\mathsf{f}}(E)$$
where $\mathsf{M}^\diamond(E)$ and $E^{<\omega}$ respectively denote the finitary multiset and the finite sequence constructions on a wqo E.

The invariants of most elementary wqos can be computed using Table 2. However some cases, for instance computing the ordinal invariants of $\mathsf{P}_{\mathsf{f}}(E)$, or computing $\mathbf{w}(E_1 \times E_2)$, need to be handled differently. To reduce the boilerplate code to a minimum, we first normalize our expressions so that we can minimize the appearance of problematic subexpressions with the help of some isomorphisms in the spirit of those described in Lemma 2.20. The rewriting rules given in Figure 5 preserve wqos modulo wpo-isomorphism, and define a strongly normalizing, confluent rewrite system on the expressions for elementary wqos.

$$P_f(\alpha) \to \alpha \,,$$
$$E \times (E_1 \sqcup E_2) \to (E \times E_1) \sqcup (E \times E_2)\,,$$
$$(E_1 \sqcup E_2) \times E \to (E_1 \times E) \sqcup (E_2 \times E)\,,$$
$$M^\diamond(E_1 \sqcup E_2) \to M^\diamond(E_1) \times M^\diamond(E_2)\,,$$
$$P_f(E_1 \sqcup E_2) \to P_f(E_1) \times P_f(E_2)\,.$$

Figure 5: Rewrite rules for elementary wqos

The normal form of an expression is computable. We therefore assume that an elementary wqo is always given via its expression in *normal form*, i.e., that cannot be rewritten by the above rules. Note that the rewrite rule $P_f(\alpha) \to \alpha$ is valid because $P_f(\alpha) \cong_{wpo} 1 + \alpha$ for any ordinal α (see Lemma 2.20), and α is indecomposable in our definition of elementary wqos.

5.1 Maximal order type and width of elementary wqos

In this section, we provide an algorithm to compute both the width and the maximal order type of an elementary wqo E. The main idea of this double computation is that most elementary wqos verify the property $\mathbf{w}(E) = \mathbf{o}(E)$, in which case the following "sandwich lemma" can be applied.

Lemma 5.2 (Powerset Sandwich)**.** *Let A be a* wqo *such that* $\mathbf{w}(A) = \mathbf{o}(A) \geq \omega$. *Then* $\mathbf{w}(P_f(A)) = \mathbf{o}(P_f(A)) = 2^{\mathbf{o}(A)}$.

Proof. According to Theorem 3.11 for the lower bound, and Theorem 3.3 for the upper bound, we have

$$2^{\mathbf{w}(A)} \leq \mathbf{w}(P_f(A)) \leq \mathbf{o}(P_f(A)) \leq 2^{\mathbf{o}(A)}\,. \qquad \square$$

Some elementary wqos do not verify $\mathbf{o}(E) = \mathbf{w}(E)$ (take for instance $E = \alpha$). Moreover, if E is a Cartesian product, we cannot easily check if $\mathbf{o}(E) = \mathbf{w}(E)$ as the width of the Cartesian product is not yet completely understood. We tackle this issue with recent work from Vialard (2024*b*), which reduces the width of a Cartesian product to its maximal order type in specific situations.

Theorem 5.3 (Vialard (2024*b*, Theorem 5.2))**.** *Let $A_1, \ldots, A_n$ be* wqos. *If there exist $i \neq j \in [1, n]$ and $\alpha, \beta > 0$ such that $\mathbf{o}(A_i) = \omega^{\omega \cdot \alpha}$ and $\mathbf{o}(A_j) = \omega^{\omega \cdot \beta}$, then $\mathbf{w}(A_1 \times \cdots \times A_n) = \mathbf{o}(A_1 \times \cdots \times A_n)$.*

As for the powerset, it will become clear in Lemma 5.4 that most products of elementary wqos satisfy the hypotheses of Theorem 5.3, and that the only exceptions for elementary wqos given in normal form are either ordinals, on which the finitary powerset construction acts trivially, or on a disjoint sum of elementary wqos, on which the finitary powerset acts as a disjoint sum of finitary powersets (see Lemma 2.20).

Lemma 5.4 (Case disjunction)**.** *Let E be an elementary* wqo *(given through its expression in normal form). Then:*
 (i) Either $\mathbf{w}(E) = \mathbf{o}(E) = \omega^{\omega \cdot \beta}$ for some ordinal $\beta > 0$,
 (ii) or $E = E_1 \sqcup E_2$ with E_1, E_2 elementary wqos,
 (iii) or $E = \alpha \geq \omega^\omega$ indecomposable,

Proof. By induction on the expression in normal form of E:

563 Case $E = \alpha$: then E satisfies (*iii*).

564 Case $E = E_1 \sqcup E_2$: then E satisfies (*ii*).

565 Case $E = E_1^{<\omega}$: Since any elementary wqo has m.o.t. > 1, E satisfies (*i*) according to Table 2.

566 Case $E = \mathsf{M}^\diamond(E_1)$: E is in normal form, hence E_1 cannot be a disjoint sum (see Figure 5). By
567   induction hypothesis E_1 satisfies either (*i*), (*ii*), or (*iii*), and we know that (*ii*) is impossible.
568   Hence $\mathbf{o}(E_1) > 1$ is additively indecomposable, therefore E satisfies (*i*), see Table 2.

569 Case $E = E_1 \times E_2$: E is in normal form, hence E_1 cannot be a disjoint sum (see Figure 5). By
570   induction hypothesis E_1 and E_2 both satisfy either (*i*) or (*iii*). Thus $\mathbf{o}(E_1)$ and $\mathbf{o}(E_2)$ are of
571   the form $\omega^{\omega \cdot \alpha}$ and $\omega^{\omega \cdot \beta}$ for some $\alpha, \beta > 0$. Therefore, according to Theorem 5.3, $\mathbf{w}(E) =$
572      $\mathbf{o}(E) = \mathbf{o}(E_1) \otimes \mathbf{o}(E_2) = \omega^{\omega \cdot (\alpha \oplus \beta)}$ hence E verifies (*i*).

573 Case $E = \mathsf{P}_f(E_1)$: E is in normal form, so E_1 cannot be a disjoint sum nor an ordinal (see
574   Figure 5). Combining this with our induction hypothesis, we know that E_1 satisfies (*i*).
575   Thus, following Lemma 5.2, $\mathbf{w}(E) = \mathbf{o}(E) = 2^{\mathbf{o}(E_1)} = \omega^{\mathbf{o}(E_1)}$ hence E satisfies (*i*). $\square$

576 We can deduce from the proof of Lemma 5.4 that for any elementary wqo E, if $E = \mathsf{P}_f(E_1)$
577 in normal form then $\mathbf{w}(E) = \mathbf{o}(E) = 2^{\mathbf{o}(E_1)}$, because the case (*ii*) is impossible due to the rewrite
578 rules of Figure 5. Similarly if E is a Cartesian product in normal form then $\mathbf{w}(E) = \mathbf{o}(E)$. Hence
579 we know the width and maximal order type of all elementary wqos, which is summarized in
580 Table 3.

$\mathbf{E}$	$\mathbf{w}(E)$	$\mathbf{o}(E)$
$\alpha, E_1^{<\omega}, E_1 \sqcup E_2$ $\mathsf{M}^\diamond(E_1)$	Table 2	Table 2
$\mathsf{P}_f(E_1)$	$2^{\mathbf{o}(E_1)}$	$2^{\mathbf{o}(E_1)}$
$E_1 \times E_2$	$\mathbf{o}(E_1) \otimes \mathbf{o}(E_2)$	$\mathbf{o}(E_1) \otimes \mathbf{o}(E_2)$

Table 3. : Width and m.o.t. for elementary wqos

5.2 Height of elementary wqos

582 Given a wqo E, its height $\mathbf{h}(E)$ is computable from the heights of its sub-expressions (see Table 2),
583 with the notable exception of $\mathbf{h}(\mathsf{P}_f(E))$ which cannot be expressed as a function of $\mathbf{h}(E)$ as proven
584 in Section 4.

585 Recall that the height $\mathbf{h}(\mathsf{P}(E))$ of the full powerset construction can be computed via
586 $\mathbf{h}(\mathsf{P}(E)) = \mathbf{o}(E) + 1$ (see Remark 3.4). The main contribution of this section, Theorem 5.11, lever-
587 ages this equality to connect the height of the finitary powerset to the supremum of the maximal
588 order type of *approximations* of E, that we will introduce in Definitions 5.5 and 5.8.

589 Approximations will be defined inductively on the parse tree of an elementary wqo, by repla-
590 cing constructors with "strict approximations." For instance, an ordinal α will be approximated by
591 any smaller ordinal $\alpha' < \alpha$, and the finitary sequence construction $E^{<\omega}$ will be approximated by
592 Cartesian products of the form $E^{\times n}$ for $n < \omega$. To approximate finite multisets, we define for any
593 wqo A and for any $n < \omega$, the set $\mathsf{M}_n^\diamond(A)$ to be the set of multisets of exactly n elements ordered
594 with the multiset embedding.

Definition 5.5 (Approximation of an elementary wqo)**.** Let E, E' be two elementary wqos. We say E' *approximates* E, denoted with $E' <_{\mathsf{approx}} E$, iff one of the following clauses holds:[h]

- $E = \alpha$ and $E' \cong_{\mathsf{wpo}} \alpha' < \alpha$,
- $E = E_1 \sqcup E_2$ and $E' \cong_{\mathsf{wpo}} E_1' \sqcup E_2'$ where $E_1' <_{\mathsf{approx}} E_1$ and $E_2' <_{\mathsf{approx}} E_2$,
- $E = E_1 \times E_2$ and $E' \cong_{\mathsf{wpo}} E_1' \times E_2'$ where $E_1' <_{\mathsf{approx}} E_1$ and $E_2' <_{\mathsf{approx}} E_2$,
- $E = E_1^{<\omega}$ and $E' \cong_{\mathsf{wpo}} (E_1')^{\times n}$ where $E_1' <_{\mathsf{approx}} E_1$ and $n < \omega$,
- $E = \mathsf{M}^{\diamond}(E_1)$ and $E' \cong_{\mathsf{wpo}} \mathsf{M}_n^{\diamond}(E_1')$ where $E_1' <_{\mathsf{approx}} E_1$ and $n < \omega$,
- $E = \mathsf{P}_{\mathsf{f}}(E_1)$ and $E' \cong_{\mathsf{wpo}} \mathsf{P}_{\mathsf{f}}(E_1')$ where $E_1' <_{\mathsf{approx}} E_1$.

The notion of approximation can be understood as a principled way of considering strictly simpler substructures of a given elementary wqo E, and the following fact ensures that approximations are indeed substructures of E, proven using a straightforward induction on its expression.

Fact 5.6. $E' <_{\mathsf{approx}} E \implies E' \leq_{\mathsf{st}} E$.

Example 5.7. *Let $E = (\omega^{\omega} \times \mathsf{P}_{\mathsf{f}}(\omega^{\omega}))^{<\omega}$. Typical approximations of E include the sets $((\omega \cdot n) \times \mathsf{P}_{\mathsf{f}}(\omega \cdot n))^{\times n}$ for $n < \omega$. Indeed, those are approximations of E, and any other approximation $E' <_{\mathsf{approx}} E$ is wpo-isomorphic to a substructure of one of those.*

The key idea behind the introduction of approximations is to introduce *smoothened* (weakened) versions of the ordinal invariants, that will behave better with respect to the finitary powerset construction. These new invariants are defined as the supremum of the corresponding invariant over all approximations.

Definition 5.8 (Weakened ordinal invariants)**.** Let $\mu \in \{\mathbf{o}, \mathbf{h}, \mathbf{w}\}$ be an ordinal invariant, and E be an elementary wqo. Then, the *weakened ordinal invariant* associated to μ is defined as $\underline{\mu}(E) \overset{\text{def}}{=} \sup_{E' <_{\mathsf{approx}} E} \{\mu(E') + 1\}$.

Let us illustrate the computation of $\underline{\mathbf{o}}(E)$ on the example from Example 5.7.

Example 5.9. *Let $E = (\omega^{\omega} \times \mathsf{P}_{\mathsf{f}}(\omega^{\omega}))^{<\omega}$ as in Example 5.7. Then, we have*

$$
\begin{aligned}
\underline{\mathbf{o}}(E) &= \sup_{n<\omega} \left\{ \mathbf{o}((\omega \cdot n) \times \mathsf{P}_{\mathsf{f}}(\omega \cdot n))^{\times n}) + 1 \right\} \\
&= \sup_{n<\omega} \left\{ (\mathbf{o}(\omega \cdot n) \otimes \mathbf{o}(\mathsf{P}_{\mathsf{f}}(\omega \cdot n)))^{\otimes n} + 1 \right\} \\
&= \sup_{n<\omega} \left\{ ((\omega \cdot n) \otimes (\omega \cdot n))^{\otimes n} + 1 \right\} \\
&= \sup_{n<\omega} \left\{ \omega^{2n} \cdot n^{2n} + 1 \right\} \\
&= \omega^{\omega},
\end{aligned}
$$

where $\alpha^{\otimes n}$ denotes the natural product $\alpha \otimes \cdots \otimes \alpha$ n times. On the other hand, using the inductive definition of the maximal order type from Table 3 on elementary wqos, we have $\mathbf{o}(E) = \omega^{\omega^{\omega \cdot 2}}$ which is much larger than $\underline{\mathbf{o}}(E)$.

As for ordinary ordinal invariants, we are going to use structural monotonicity on a regular basis, which is ensured by the following lemma that extends Lemma 2.18 to approximations.

---

[h]The recursion in the definition is well-founded so the relation $<_{\mathsf{approx}}$ is well defined.

Fact 5.10. *Let E_1, E_2 be elementary* wqos *such that the normal form of E_1 is a sub-expression of the normal form of E_2, and let $\mu \in \{\mathbf{o}, \mathbf{h}, \mathbf{w}\}$. Then $\underline{\mu}(E_1) \leq \underline{\mu}(E_2)$.*

Proof. If E_1 is a sub-expression of E_2, then E_1 is isomorphic to a substructure of E_2. Furthermore, for any $E_1' <_{\mathsf{approx}} E_1$, there exists $E_2' <_{\mathsf{approx}} E_2$ such that $E_1' \leq_{\mathsf{st}} E_2'$. Hence $\mu(E_1') \leq \mu(E_2')$ according to Lemma 2.18, therefore $\underline{\mu}(E_1) \leq \underline{\mu}(E_2)$. □

We are now ready to state the main technical result of this section, stating that computing $\underline{\mathbf{o}}(E)$ is sufficient to compute $\mathbf{h}(\mathsf{P}_{\mathsf{f}}(E))$.

Theorem 5.11. *For every elementary* wqo *E, $\mathbf{h}(\mathsf{P}_{\mathsf{f}}(E)) = \underline{\mathbf{o}}(E)$.*

From this theorem and Example 5.9, one concludes that the finitary powerset of the elementary wqo E from Example 5.7 has height $\mathbf{h}(\mathsf{P}_{\mathsf{f}}(E)) = \underline{\mathbf{o}}(E) = \omega^\omega$.

Computing $\underline{\mathbf{o}}(E)$. Before proving Theorem 5.11, let us first show how to compute $\underline{\mathbf{o}}(E)$ for any elementary wqo E, that will make the situation of Example 5.9 systematic. The rules for computing $\underline{\mathbf{o}}(E)$ are summarized in Table 4, where some hypotheses are required to ensure the correctness of the computations. After establishing these rules in Lemma 5.12, we will show in Lemma 5.15 that these hypotheses are always satisfied for elementary wqos.

E	$\underline{\mathbf{o}}(E)$	**Hypothesis**
$E_1 \times E_2$	$\max(\underline{\mathbf{o}}(E_1), \underline{\mathbf{o}}(E_2))$	$\underline{\mathbf{o}}(E_1), \underline{\mathbf{o}}(E_2)$ indecomposable
$E_1 \sqcup E_2$	$\max(\underline{\mathbf{o}}(E_1), \underline{\mathbf{o}}(E_2))$	$\underline{\mathbf{o}}(E_1), \underline{\mathbf{o}}(E_2)$ indecomposable
$\mathsf{M}^\diamond(E_1)$	$\underline{\mathbf{o}}(E_1)$	$\underline{\mathbf{o}}(E_1)$ indecomposable
$E_1^{<\omega}$	$\underline{\mathbf{o}}(E_1)$	$\underline{\mathbf{o}}(E_1)$ indecomposable
$\mathsf{P}_{\mathsf{f}}(E_1)$	$2^{\underline{\mathbf{o}}(E_1)}$	$\underline{\mathbf{o}}(E_1) = \underline{\mathbf{w}}(E_1)$

Table 4. : Computing the weakened maximal order type under conditions.

Lemma 5.12. *For E elementary, $\underline{\mathbf{o}}(E)$ can be computed as described in Table 4 in cases following the given hypotheses.*

Proof. Let E_1, E_2 be elementary wqos, let $\alpha = \underline{\mathbf{o}}(E_1)$, and let $\beta = \underline{\mathbf{o}}(E_2)$

- Without loss of generality, let us assume that $\beta \leq \alpha$. Then, with the hypothesis that α and β are multiplicatively indecomposable:
  - $\underline{\mathbf{o}}(E_1 \sqcup E_2) \leq \sup\{\, \alpha' \oplus \beta' + 1 \mid \alpha' < \alpha, \beta' < \beta \,\} = \alpha$.
  - $\underline{\mathbf{o}}(E_1 \times E_2) \leq \sup\{\, \alpha' \otimes \beta' + 1 \mid \alpha' < \alpha, \beta' < \beta \,\} = \alpha$.
  - $\underline{\mathbf{o}}(E_1^{<\omega}) \leq \sup\{\, (\alpha')^{\otimes n} + 1 \mid \alpha' < \alpha, n < \omega \,\} \leq \alpha$
  - $\mathsf{M}_{\mathsf{n}}^\diamond(A) \geq_{\mathsf{aug}} (A)^{\times n}$ for any wqo A, hence $\underline{\mathbf{o}}(\mathsf{M}^\diamond(E_1)) \leq \underline{\mathbf{o}}(E_1^{<\omega}) = \alpha$.

  Observe that in these four cases, E_1 is a sub-expression of some E with $\alpha \leq \underline{\mathbf{o}}(E)$ according to Fact 5.10.

- $2^{\underline{\mathbf{w}}(E_1)} \leq \underline{\mathbf{w}}(\mathsf{P_f}(E_1)) \leq \underline{\mathbf{o}}(\mathsf{P_f}(E_1)) \leq 2^{\underline{\mathbf{o}}(E_1)}$ according to Theorems 3.3 and 3.11. Hence if $\underline{\mathbf{o}}(E_1) = \underline{\mathbf{w}}(E_1)$, then $\underline{\mathbf{o}}(\mathsf{P_f}(E_1)) = 2^{\underline{\mathbf{o}}(E_1)}$. □

The condition $\underline{\mathbf{o}}(E_1) = \underline{\mathbf{w}}(E_1)$ for $E = \mathsf{P_f}(E_1)$ in Table 4 is not very restrictive, since operations like $E^{<\omega}$ and $\mathsf{M}^{\diamond}(E)$ do not drastically increase the weakened m.o.t. of an elementary wqo as they do for the m.o.t., to the point that $\underline{\mathbf{o}} = \underline{\mathbf{w}}$ holds for most elementary wqos. To prove that, we develop some lower bounds on $\underline{\mathbf{w}}$.

One of the difficulty in computing the weakened width will come from the finitary sequence construction and the finite multiset construction, that will essentially require us to compute the width of structures of the form $\mathsf{M}_n^{\diamond}(\alpha)$. As we will see, it suffices for our purposes to focus on $\mathsf{M}_2^{\diamond}(\alpha)$ to obtain our desired lower bounds on $\underline{\mathbf{w}}$.

Definition 5.13. For any ordinal α, let us define $\mathsf{Inc}_\alpha \stackrel{\text{def}}{=} \mathsf{M}_2^{\diamond}(\alpha)$.

Remark that Inc_α can be seen as the set of pairs of ordinals (β_0, β_1) such that $\beta_0 \leq \beta_1 < \alpha$, ordered pointwise, and therefore corresponds to a substructure of the Cartesian product $\alpha \times \alpha$. Furthermore, Inc_α reflects into $\mathsf{P_f}(\alpha \times 2)$ through the function $(\beta_0, \beta_1) \mapsto \{(\beta_1, 0), (\beta_0, 1)\}$.

Let us now establish a lower bound on $\underline{\mathbf{w}}(\mathsf{Inc}_\alpha)$, which is done by extending the definition of $\underline{\mathbf{w}}$ to well-quasi-orders of the form $\mathsf{M}_2^{\diamond}(\alpha)$ as follows: $E' <_{\mathsf{approx}} \mathsf{M}_2^{\diamond}(\alpha)$ iff $E' \cong_{\mathsf{wpo}} \mathsf{M}_2^{\diamond}(\alpha')$ for some $\alpha' < \alpha$. As a consequence of this definition, we always have $\underline{\mathbf{w}}(\mathsf{M}^{\diamond}(\alpha)) \geq \underline{\mathbf{w}}(\mathsf{Inc}_\alpha)$.

> **for isa:** Check carefully this lemma and its proof.

Lemma 5.14. *Let $\alpha \geq \omega^\omega$ be an indecomposable ordinal. Then,*

$$\underline{\mathbf{w}}(\mathsf{Inc}_\alpha) = \sup_{\alpha' < \alpha} \mathbf{w}(\mathsf{Inc}_{\alpha'}) \geq \alpha \,.$$

Proof. For all $\omega \leq \alpha' < \alpha$, let us define the map $f \colon \mathsf{Dec}(\alpha' - \omega) \to \mathsf{Inco}(\mathsf{Inc}_{\alpha'})$ which to any decreasing sequence $\alpha' - \omega > \beta_0 > \beta_1 > \ldots$ associates the antichain $((k, \beta_k + \omega))_{k < \omega}$. Therefore $\underline{\mathbf{w}}(\mathsf{Inc}_{\alpha'}) = \mathbf{h}(\alpha' - \omega) = \alpha' - \omega$. Hence $\underline{\mathbf{w}}(\mathsf{Inc}_\alpha) = \sup_{\alpha' < \alpha} \alpha' - \omega = \alpha - \omega = \alpha$ since α is indecomposable.

□

As we did with Lemma 5.4, let us present a structural lemma which reveals in which cases $\underline{\mathbf{w}} \neq \underline{\mathbf{o}}$.

Lemma 5.15 (Case disjunction). *Let E be an elementary wqo (given through its expression in normal form). Then, $\underline{\mathbf{o}}(E)$ is indecomposable, and one of the following holds:*

(i) $\underline{\mathbf{w}}(E) = \underline{\mathbf{o}}(E)$,

(ii) $E = E_1 \sqcup E_2$ a disjoint union of elementary wqos,

(iii) $E = \alpha$ an ordinal.

Proof. Observe that for any elementary wqo E, $\underline{\mathbf{w}}(E) \leq \underline{\mathbf{o}}(E)$ because $\mathbf{w}(E') \leq \mathbf{o}(E')$ for any $E' <_{\mathsf{approx}} E$. Therefore, to conclude that $\underline{\mathbf{o}}(E) = \underline{\mathbf{w}}(E)$, it suffices to prove that $\underline{\mathbf{w}}(E) \geq \underline{\mathbf{o}}(E)$. Let us now proceed by induction on the expression of E, but before that, let us point out that aside from $\mathsf{P_f}$, the weakened maximal order type of every elementary operation can be computed under an indecomposability hypothesis. Moreover Table 4 shows how the indecomposability of the weakened m.o.t. is preserved by our constructions.

Case $E = \alpha$ indecomposable satisfies *(iii)*, and we can check that $\underline{\mathbf{o}}(\alpha) = \alpha$ is indecomposable.

688   Case $E = \mathsf{P_f}(E_1)$: By induction hypothesis E_1 verifies (i) since E is in normal form, hence cannot
689   be an ordinal nor a disjoint union from the rules of Figure 5. Therefore $\underline{\mathbf{w}}(E) = \underline{\mathbf{o}}(E) =$
690      $2^{\underline{\mathbf{o}}(E_1)}$ applying Table 4. Hence E verifies (i). Finally, because $\underline{\mathbf{o}}(E_1)$ is indecomposable
691   according to the induction hypothesis, we conclude that $\underline{\mathbf{o}}(E) = 2^{\underline{\mathbf{o}}(E_1)}$ is indecomposable as
692   well.
693   Case $E = E_1 \sqcup E_2$ satisfies (ii).
694   Case $E = E_1 \times E_2$: Assume that $\underline{\mathbf{o}}(E_1) = \alpha \geq \beta = \underline{\mathbf{o}}(E_2)$, and therefore that $\underline{\mathbf{o}}(E) = \alpha$ (see
695   Table 4). Now there exists $E_2' <_{\mathsf{approx}} E_2$ such that $\omega \leq_{\mathsf{st}} E_2'$. Thus for any $E_1' <_{\mathsf{approx}} E_1$,
696   one can leverage the structural monotonicity of the width (Lemma 2.18) and the specific
697   inequality between width and m.o.t. of Lemma 3.10 to obtain:

$$\mathbf{w}(E_1' \times E_2') \geq \mathbf{w}(E_1' \times \omega) \geq \mathbf{o}(E_1')$$

698   We conclude that $\underline{\mathbf{w}}(E) \geq \underline{\mathbf{o}}(E_1) = \underline{\mathbf{o}}(E)$, proving that E verifies (i).
699   Case $E = \mathsf{M}^{\diamond}(E_1)$: For all $E_1' <_{\mathsf{approx}} E_1$, we know that $\mathsf{M}_2^{\diamond}(E_1') <_{\mathsf{approx}} E$ by definition of ap-
700   proximation. Since $E_1' \leq_{\mathsf{aug}} \mathbf{o}(E_1')$, and because finite multisets are monotonic with respect
701   to augmentations, we have $\mathsf{M}_2^{\diamond}(E_1') \leq_{\mathsf{aug}} \mathsf{M}_2^{\diamond}(\mathbf{o}(E_1')) = \mathsf{Inc}_{\mathbf{o}(E_1')}$. As a consequence, and
702   according to Lemma 5.14:

$$
\begin{aligned}
\underline{\mathbf{w}}(E) &\geq \sup_{E_1' <_{\mathsf{approx}} E_1} \left\{ \mathbf{w}(\mathsf{Inc}_{\mathbf{o}(E_1')}) \right\} \\
&\geq \underline{\mathbf{w}}(\mathsf{Inc}_{\mathbf{o}(E_1)}) \\
&\geq \underline{\mathbf{o}}(E_1)
\end{aligned}
$$

703   Furthermore, $\underline{\mathbf{o}}(E) = \underline{\mathbf{o}}(E_1)$ according to Table 4, and we have proven that E satisfies (i).
704   Case $E = E_1^{<\omega}$: For any $E_1' <_{\mathsf{approx}} E_1$, for any $n < \omega$, $\mathsf{M}_n^{\diamond}(E_1') \geq_{\mathsf{aug}} (E_1')^{\times n}$, thus $\underline{\mathbf{w}}(E) \geq$
705      $\underline{\mathbf{w}}(\mathsf{M}^{\diamond}(E_1))$. Since $\underline{\mathbf{o}}(E) = \underline{\mathbf{o}}(E_1) = \underline{\mathbf{o}}(\mathsf{M}^{\diamond}(E_1))$ according to Table 4, and by the previous
706   case, we have $\underline{\mathbf{w}}(E) \geq \underline{\mathbf{o}}(E)$. Hence E verifies (i).
707   □

708   Hence, the hypotheses in Table 4 are verified for all elementary wqo E, and assuming that E
709 is in normal form, we can inductively compute $\underline{\mathbf{o}}(E)$ without any hypothesis, as summarized in
710 Table 5.

E **in normal form**	$\mathbf{h}(\mathsf{P_f}(E)) = \underline{\mathbf{o}}(E)$
α	α
$E_1 \times E_2$	$\max(\underline{\mathbf{o}}(E_1), \underline{\mathbf{o}}(E_2))$
$\mathsf{M}^{\diamond}(E_1)$	$\underline{\mathbf{o}}(E_1)$
$E_1^{<\omega}$	$\underline{\mathbf{o}}(E_1)$
$\mathsf{P_f}(E_1)$	$2^{\underline{\mathbf{o}}(E_1)}$

Table 5. : Computing the weakened maximal order type inductively.

711   **Theorem 5.16.** $\mathbf{h}(E)$ *is computable for elementary E given by its elementary expression.*

712   *Proof.* Assume E is in normal form, and see Tables 2 and 5.   □

Proving Theorem 5.11. It remains for us to prove the main theorem of this section, that we restate hereafter for readability.

Theorem 5.11. *For every elementary* wqo E, $\mathbf{h}(\mathsf{P}_f(E)) = \underline{\mathbf{o}}(E)$.

The easiest inequality to obtain is the lower bound $\underline{\mathbf{o}}(E) \leq \mathbf{h}(\mathsf{P}_f(E))$. It can be dealt with through properties of ideal completions. Recall that an *ideal* I of a wqo A is a non-empty downwards closed and up-directed subset of A, see, e.g., Goubault-Larrecq et al. (2020). For completeness, let us recall that a set I being *up-directed* means that for every $a, b \in I$, there exists $c \in I$ such that $a \leq c$ and $b \leq c$.

Definition 5.17 (Ideal Completion)**.** The set of ideals of A ordered with Hoare's embedding is denoted $\mathsf{Idl}(A)$, and is called the *ideal completion* of A.[i]

The goal of this article is not to study ideal completions in depth, and we will only use some basic properties that were already established in prior work Halfon (2018).

Fact 5.18 (See Halfon (2018))**.** *The following properties hold for any* wqo A, B *and ordinal* α:
(1) Ideal completion commutes with $\sqcup$, $\times$, $\mathsf{M}_n^\diamond$ *and* P_f, *i.e.,* $\mathsf{Idl}(A \sqcup B) \cong \mathsf{Idl}(A) \sqcup \mathsf{Idl}(B)$, *etc.,*
(2) $\mathsf{Idl}(\alpha) \cong \alpha + 1$,
(3) $\mathsf{P}_f(\mathsf{Idl}(A)) \cong \mathsf{P}(A)$.

With point (3) above we see that $\mathsf{Idl}(A)$ is not always a wqo, as $\mathsf{P}(A)$ may not be one! However, since an approximation E' of an elementary E only uses constructions that commute with ideal completion, we get Fact 5.19. This fact not only states that the ideal completion of an approximation is again a wqo, but that it even remains an approximation of the original elementary wqo. This will be particularly useful, because $\mathbf{h}(\mathsf{P}(E)) = \mathbf{o}(E) + 1$ for any wqo E, as we saw in Remark 3.4.

Fact 5.19. *Assume* $E' <_{\mathsf{approx}} E$ *for an elementary* wqo E. *Then* $\mathsf{Idl}(E') <_{\mathsf{approx}} E$.

Lemma 5.20. *For every elementary* wqo E, $\underline{\mathbf{o}}(E) \leq \mathbf{h}(\mathsf{P}_f(E))$.

Proof. Let $E' <_{\mathsf{approx}} E$. Then $\mathsf{Idl}(E') <_{\mathsf{approx}} E$ thanks to Fact 5.19, and in particular $\mathsf{Idl}(E') \leq_{\mathsf{st}} E$ (Fact 5.6), thus $\mathsf{P}_f(\mathsf{Idl}(E')) \leq_{\mathsf{st}} \mathsf{P}_f(E)$.

Using the monotonicity of the height with respect to substructures (Lemma 2.18), we conclude that $\mathbf{h}(\mathsf{P}_f(E)) \geq \mathbf{h}(\mathsf{P}_f(\mathsf{Idl}(E')))$. Now, by Fact 5.18, we know that $\mathsf{P}_f(\mathsf{Idl}(E')) \cong \mathsf{P}(E')$. Finally, leveraging Remark 3.4, we obtain $\mathbf{h}(\mathsf{P}_f(\mathsf{Idl}(E'))) = \mathbf{h}(\mathsf{P}(E')) = \mathbf{o}(E') + 1$. Hence, we have proven that $\mathbf{h}(\mathsf{P}_f(E)) \geq \underline{\mathbf{o}}(E)$. $\qquad\square$

The upper bound $\mathbf{h}(\mathsf{P}_f(E)) \leq \underline{\mathbf{o}}(E)$, in contrast, is quite subtle to prove. Observe that *augmentations* (Definition 2.14) do not preserve the height, depriving us of one of our favorite ways to prove bounds. That is why we introduce the notion of *condensation*.

Definition 5.21 (Condensation)**.** A mapping $f \colon A \to B$ between two wqos is a *condensation* if it is surjective, monotonic, and whenever $b \leq_B f(y)$, there exists $x \leq_A y$ such that $b = f(x)$. When there exists a condensation from A to B, we note $B \leq_{\mathsf{cond}} A$.

It appears that condensations are also known in the literature as *p-morphisms* in the context of Kripke frames and modal logic. In the specific context of quasi-orders, it seems that it is less

---

[i]When A is a wpo, ordering ideals by inclusion or by Hoare's embedding is equivalent.

standard than reflection, augmentation, and substructure. Intuitively, a condensation can be seen as a quotient, where the quotient ordering and the original ordering are related.

Example 5.22. *The function* $\iota\colon A^{\times n} \to \mathsf{M}_{\mathsf{n}}^{\diamond}(A)$ *mapping* $(a_1, \ldots, a_n)$ *to the multiset* $\{\!| a_1, \ldots, a_n |\!\}$ *is a condensation.*

Remark 5.23. A surjective monotonic function $f\colon A \to B$ is a condensation if and only if the image of any downwards-closed set is itself downwards-closed. In topological terms, f is continuous, closed, and surjective.

The key property of condensations from A to B is that one can simulate decreasing sequences $(b_i)_{i\in\mathbb{N}}$ occurring in B through a careful selection of pre-images $a_i \in f^{-1}(b_i)$. As a consequence, the height of B is bounded by the height of A.

Lemma 5.24. *If* $B \leq_{\mathsf{cond}} A$, *then* $\mathbf{h}(B) \leq \mathbf{h}(A)$.

Proof. Let $f\colon A \to B$ be a condensation, $b_1 > \cdots > b_{n+1}$ a strictly decreasing sequence in B, and $a_1 > \cdots > a_n$ a strictly decreasing sequence in A such that $f(a_i) = b_i$ for all $i \leq n$.

Since $b_{n+1} \leq b_n$ there exists $a_{n+1} \leq a_n$ such that $f(a_{n+1}) = b_{n+1}$. Since f is monotonic, $b_{n+1} \not\geq b_n$ implies that $a_{n+1} \not\geq a_n$. Hence $a_1 > \cdots > a_{n+1}$. This shows that with any strictly decreasing sequence of B we can associate a strictly decreasing sequence of A in a way that respects prefix order, hence $\mathsf{Dec}(B)$ is a substructure of $\mathsf{Dec}(A)$ modulo isomorphism. We have proven that $\mathbf{h}(B) \leq \mathbf{h}(A)$. $\qquad\square$

Example 5.25. *The function* $\iota\colon \mathsf{M}^{\diamond}(A) \to \mathsf{P}_{\mathsf{f}}(A)$ *mapping a multiset M to the set* $\{\, a \mid a \in M \,\}$ *is monotonic, surjective, but is not always a condensation.*

Proof. Take $A = 3$. In $\mathsf{P}_{\mathsf{f}}(A)$ one has $\{0, 1\} \sqsubseteq_{\mathscr{H}} \{2\}$, but no multiset M in $\iota^{-1}(\{0, 1\})$ is dominated by $\{\!| 2 |\!\}$ since any such M contains at least one 0 and one 1. More generally, we know of instances where $\mathbf{h}(\mathsf{M}^{\diamond}(A)) < \mathbf{h}(\mathsf{P}_{\mathsf{f}}(A))$ (see family $\mathscr{P}_\alpha$ in Section 4.2). $\qquad\square$

We now establish two properties of condensations. First we show that the constructors used to build approximated wqos are monotonic with respect to $\leq_{\mathsf{cond}}$ (Fact 5.27). Then we show that approximations can be extended while respecting $\leq_{\mathsf{cond}}$ (Lemma 5.28). We will actually consider a variation $\leq_{\mathsf{cond+st}}$ of $\leq_{\mathsf{cond}}$, that we formally state hereafter.

Definition 5.26. We define $\leq_{\mathsf{cond+st}}$ as the transitive closure of the relations $\leq_{\mathsf{cond}}$ and $\leq_{\mathsf{st}}$, which is a quasi-ordering over wqos.

Fact 5.27 (Monotonicity)**.** *The operations* $\sqcup$, $\times$, P_{f}, $\mathsf{M}^{\diamond}$, $^{<\omega}$ *and* $+$ *are monotonic with respect to* $\leq_{\mathsf{cond}}$.

Lemma 5.28. *Let E be an elementary* wqo *and* $E' <_{\mathsf{approx}} E$. *Then there exists* $E'' <_{\mathsf{approx}} E$ *such that* $1 + E' \leq_{\mathsf{cond+st}} E''$.

Proof. By induction on the expression of E in normal form. First observe that:

Case $E = \alpha$: for all $\alpha' < \alpha$, $1 + \alpha' < \alpha$ since α is infinite.
Case $E = E_1 \sqcup E_2$: $1 + (E_1' \sqcup E_2') \leq_{\mathsf{cond}} (1 + E_1') \sqcup (1 + E_2')$ for all $E_1' <_{\mathsf{approx}} E_1$, $E_2' <_{\mathsf{approx}}$
  E_2. Here the condensation function maps the two minimal elements of $(1 + E_1') \sqcup (1 + E_2')$
  to the unique minimal element of $1 + (E_1' \sqcup E_2')$.

Case $E = E_1 \times E_2$: $1 + (E_1' \times E_2') \leq_{\text{st}} (1 + E_1') \times (1 + E_2')$.

Case $E = E_1^{<\omega}$: $1 + (E_1')^{\times n} \leq_{\text{st}} (1 + E_1')^{\times n}$.

Case $E = \mathsf{M}^\diamond(E_1)$: $1 + \mathsf{M}_n^\diamond(E_1') \leq_{\text{st}} \mathsf{M}_n^\diamond(1 + E_1')$.

Case $E = \mathsf{P}_{\mathsf{f}}(E_1)$: $1 + \mathsf{P}_{\mathsf{f}}(E_1') \leq_{\text{st}} \mathsf{P}_{\mathsf{f}}(1 + E_1')$.

By induction hypothesis, for any E_i' with $i \in \{1, 2\}$ appearing in the cases above, there exists $E_i'' <_{\text{approx}} E_i$ such that $1 + E_i' \leq_{\text{cond+st}} E_i''$. We conclude using Fact 5.27. □

Lemma 5.29. *Let E be an* elementary *wqo. For all $S \in \mathsf{P}_{\mathsf{f}}(E)$, there exists $E' <_{\text{approx}} E$ such that $\downarrow_E S \leq_{\text{cond+st}} E'$.*

Proof. By induction on the expression in normal form of E:

Case $E = \alpha$: Let $\gamma = \max\{ x \in S \}$. Then $\downarrow S \cong \gamma <_{\text{approx}} E$.

Case $E = E_1 \sqcup E_2$: $\downarrow_E S \cong \downarrow_{E_1} S_1 \sqcup \downarrow_{E_2} S_2$ where $S_1 = S \cap E_1$ and $S_2 = S \cap E_2$. By induction hypothesis there exist $E_1', E_2' <_{\text{approx}} E_1, E_2$ such that $\downarrow_{E_1} S_1 \leq_{\text{cond+st}} E_1', \downarrow_{E_2} S_2 \leq_{\text{cond+st}} E_2'$. Hence,

$$\downarrow_E S \leq_{\text{cond+st}} E_1' \sqcup E_2' \,.$$

Case $E = E_1 \times E_2$: $\downarrow_E S \leq_{\text{st}} \downarrow_{E_1} S_{|E_1} \times \downarrow_{E_2} S_{|E_2}$. By induction hypothesis there exist $E_1', E_2' <_{\text{approx}} E_1, E_2$ such that $\downarrow_{E_1} S_{|E_1} \leq_{\text{cond+st}} E_1'$ and $\downarrow_{E_2} S_{|E_2} \leq_{\text{cond+st}} E_2'$. Hence,

$$\downarrow_E S \leq_{\text{cond+st}} E_1' \times E_2' \,.$$

Case $E = E_1^{<\omega}$: Let n be the maximal length of words in S, and $S' \in \mathsf{P}_{\mathsf{f}}(E_1)$ the set of letters in words of S. Then $\downarrow_E S \leq_{\text{st}} (\downarrow_{E_1} S')^{\leq n}$. Observe that $(\downarrow_{E_1} S')^{\leq n} \leq_{\text{cond}} (1 + \leq^{E_1} S')^{\times n}$: indeed elements of $(1 + \downarrow_{E_1} S')^{\times n}$ can be seen as words in $(\downarrow_{E_1} S')^{\leq n}$ padded with extra bottom elements, with the condensation function removing the padding. By induction hypothesis, there exists $E_1' <_{\text{approx}} E_1$ such that $\downarrow_{E_1} S' \leq_{\text{cond+st}} E_1'$.

Thanks to Lemma 5.28, there exists some $E_1'' <_{\text{approx}} E_1$ such that

$$\downarrow_E S \leq_{\text{cond+st}} (1 + E_1')^{\times n} \leq_{\text{cond+st}} (E_1'')^{\times n} \,.$$

Case $E = \mathsf{M}^\diamond(E_1)$: Let n be the maximal cardinal of multisets in S, and $S' \in \mathsf{P}_{\mathsf{f}}(E_1)$ the set of elements in multisets of S. Then $\downarrow_E S \leq_{\text{st}} \mathsf{M}^\diamond_{\leq n}(\downarrow_{E_1} S') \cong \mathsf{M}_n^\diamond(1 + \downarrow_{E_1} S')$. By induction hypothesis, there exists some $E_1' <_{\text{approx}} E_1$ such that $\downarrow_{E_1} S' \leq_{\text{cond+st}} E_1'$.

Therefore, there exists some $E_1'' <_{\text{approx}} E_1$ given by Lemma 5.28, such that

$$\downarrow_E S \leq_{\text{cond+st}} \mathsf{M}_n^\diamond(1 + E_1') \leq_{\text{cond+st}} \mathsf{M}_n^\diamond(E_1'') \,.$$

Case $E = \mathsf{P}_{\mathsf{f}}(E_1)$: Let $S' = \{ x \in E_1 \mid \exists y \in S \text{ s.t. } x \in y \} \in \mathsf{P}_{\mathsf{f}}(E_1)$. Then $\downarrow_E S \leq_{\text{st}} \mathsf{P}_{\mathsf{f}}(\downarrow_{E_1} S')$. Using the induction hypothesis, there exists $E_1' <_{\text{approx}} E_1$ such that $\downarrow_{E_1} S' \leq_{\text{cond+st}} E_1'$, hence

$$\downarrow_E S \leq_{\text{st}} \mathsf{P}_{\mathsf{f}}(E_1') \,. \qquad \qquad \square$$

Lemma 5.30. $\mathbf{h}(\mathsf{P}_{\mathsf{f}}(E)) \leq \underline{\mathbf{o}}(E)$.

Proof. Recall that, by Eq. (Dh),

$$\mathbf{h}(\mathsf{P}_{\mathsf{f}}(E)) = \sup_{S \in \mathsf{P}_{\mathsf{f}}(E)} \mathbf{h}(\mathsf{P}_{\mathsf{f}}(E)_{<S}) + 1 \,.$$

Notice that, given $S \in \mathsf{P}_{\mathsf{f}}(E)$, we have $\mathbf{h}(\mathsf{P}_{\mathsf{f}}(E)_{<S}) + 1 = \mathbf{h}(\mathsf{P}_{\mathsf{f}}(E)_{\leq S})$ and $\mathsf{P}_{\mathsf{f}}(E)_{\leq S} \leq_{\text{st}} \mathsf{P}_{\mathsf{f}}(\downarrow_E S)$.

Using Lemma 5.29, there exists $E' <_{\text{approx}} E$ such that $\downarrow_E S \leq_{\text{cond+st}} E'$. As a consequence, $\mathbf{h}(\mathsf{P}_{\mathsf{f}}(E)_{\leq S}) \leq \mathbf{h}(\mathsf{P}_{\mathsf{f}}(E')) \leq \mathbf{h}(\mathsf{P}(E')) \leq \mathbf{o}(E') + 1$ according to Fact 5.18.

Therefore, for every $S \in \mathsf{P_f}(E)$, $\mathbf{h}(\mathsf{P_f}(E)_{<S} + 1) \leq \underline{\mathbf{o}}(E)$, which implies by Eq. (Dh) that $\mathbf{h}(\mathsf{P_f}(E)) \leq \underline{\mathbf{o}}(E)$. □

Combining Lemmas 5.20 and 5.30 proves Theorem 5.11: $\mathbf{h}(\mathsf{P_f}(E)) = \underline{\mathbf{o}}(E)$.

5.3 Summary

Table 6 summarizes how to compute the ordinal invariants of elementary wqos expressed in normal form. We redirect the reader to Table 5 when it comes to computing $\underline{\mathbf{o}}(E_1)$.

E	$\mathbf{h}(E)$	$\mathbf{o}(E)$	$\mathbf{w}(E)$
$\alpha, E_1 \sqcup E_2,$ $\mathsf{M}^\diamond(E_1), E_1^{<\omega}$		see Table 2	
$E_1 \times E_2$	$\mathbf{h}(E_1) \,\hat{\oplus}\, \mathbf{h}(E_2)$	$\mathbf{o}(E_1) \otimes \mathbf{o}(E_2)$	$\mathbf{o}(E_1) \otimes \mathbf{o}(E_2)$
$\mathsf{P_f}(E_1)$	$\underline{\mathbf{o}}(E_1)$	$2^{\mathbf{o}(E_1)}$	$2^{\mathbf{o}(E_1)}$

Table 6. : How to compute ordinal invariants of elementary wqos. We assume that E is expressed in normal form.

6. Conclusion

We studied the ordinal invariants of the finitary powerset of a wqo. We showed these invariants could not be expressed as functions in the ordinal invariants of the underlying wqo, and provided tight monomorphic bounds.

Even though these upper and lower bounds are often different, in Section 5, we managed to measure these invariants exactly by restricting ourselves to a well-behaved family of wqos obtained through classical operations. This part relies on new tools, e.g., we introduced weakened variations of the usual ordinal invariants, which in turn, have now opened many new questions.

Let us now propose a few avenues of research in continuity of this work.

Invariants for non-wqos. Width and height can more generally be defined over spaces that are not wqos. On these spaces, the bounds we established do not hold in general. For instance the height of $\mathsf{P_f}(A)$ when A is well-founded can reach the $\omega^{\mathbf{h}(A)}$ upper bound.

Intrinsic Weakened Invariants. The current definition of $\underline{\mathbf{o}}$, $\underline{\mathbf{w}}$, and $\underline{\mathbf{h}}$ rely on the definition of *approximation* that is defined inductively on the syntactic construction of the elementary wqo. Could we generalize the notion of approximation to arbitrary wqos? Do the weakened ordinal invariants allow us to recover functionality even outside of elementary wqos?

Elementary wqos. In what directions can we extend our family of elementary wqos while maintaining the computability of the ordinal invariants?

References

Abdulla, P. A., Čerāns, K., Jonsson, B., and Tsay, Y.-K. 2000. Algorithmic analysis of programs with well quasi-ordered domains. *Information & Computation*, 160(1–2):109–127.

Abraham, U. 1987. A note on Dilworth's theorem in the infinite case. *Order*, 4(2):107–125.

Abraham, U. and Bonnet, R. 1999. Hausdorff's theorem for posets that satisfy the finite antichain property. *Fund. Math.*, 159(1):51–69.

Abriola, S., Figueira, S., and Senno, G. 2015. Linearizing well quasi-orders and bounding the length of bad sequences. *Theor. Comp. Sci.*, 603:3–22.

Abriola, S., Schmitz, S., and Schnoebelen, Ph. 2016. On the maximal order type of well-quasi-ordered finitary powersets. Unpublished draft manuscript.

Altman, H. J. 2017. Intermediate arithmetic operations on ordinal numbers. *Mathematical Logic Quarterly*, 63(3–4):228–242.

Balasubramanian, A. R. 2020. Complexity of controlled bad sequences over finite sets of $\mathbb{N}^d$. In *Proc. 35th ACM/IEEE Symp. Logic in Computer Science (LICS 2020), Saarbrücken, Germany, July 2020*, pp. 130–140. ACM Press.

Blass, A. and Gurevich, Y. 2008. Program termination and well partial orderings. *ACM Trans. Computational Logic*, 9(3):1–26.

Bonnet, R., Finkel, A., Haddad, S., and Rosa-Velardo, F. 2013. Ordinal theory for expressiveness of well-structured transition systems. *Information & Computation*, 224:1–22.

de Jongh, D. H. J. and Parikh, R. 1977. Well-partial orderings and hierarchies. *Indag. Math.*, 39(3):195–207.

Dershowitz, N. and Manna, Z. 1979. Proving termination with multiset orderings. *Communications of the ACM*, 22(8):465–476.

Džamonja, M., Schmitz, S., and Schnoebelen, Ph. 2020. On ordinal invariants in well quasi orders and finite antichain orders. In *Well Quasi-Orders in Computation, Logic, Language and Reasoning*, volume 53 of *Trends in Logic*, chapter 2, pp. 29–54. Springer.

Figueira, D., Figueira, S., Schmitz, S., and Schnoebelen, Ph. 2011. Ackermannian and primitive-recursive bounds with Dickson's lemma. In *Proc. 26th IEEE Symp. Logic in Computer Science (LICS 2011), Toronto, Canada, June 2011*, pp. 269–278. IEEE Comp. Soc. Press.

Finkel, A. and Schnoebelen, P. 2001. Well-structured transition systems everywhere. *Theoretical Computer Science*, 256(1–2):63–92.

Fraïssé, R. 1986. *Theory of Relations*, volume 118 of *Studies in Logic and the Foundations of Mathematics*. North Holland.

Goubault-Larrecq, J., Halfon, S., Karandikar, P., Narayan Kumar, K., and Ph. Schnoebelen 2020. The ideal approach to computing closed subsets in well-quasi-orderings. In *Well Quasi-Orders in Computation, Logic, Language and Reasoning*, volume 53 of *Trends in Logic*, chapter 3, pp. 55–105. Springer.

Haddad, S., Schmitz, S., and Schnoebelen, Ph. 2012. The ordinal-recursive complexity of timed-arc Petri nets, data nets, and other enriched nets. In *Proc. 27th IEEE Symp. Logic in Computer Science (LICS 2012), Dubrovnik, Croatia, June 2012*, pp. 355–364. IEEE Comp. Soc. Press.

Halfon, S. 2018. *On Effective Representations of Well Quasi-Orderings*. Thèse de doctorat, École Normale Supérieure Paris-Saclay, France.

Higman, G. 1952. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc. (3)*, 2(7):326–336.

Kříž, I. and Thomas, R. 1990. Ordinal types in Ramsey theory and well-partial-ordering theory. In **Nešetřil, J. and Rödl, V.**, editors, *Mathematics of Ramsey Theory*, volume 5 of *Algorithms and Combinatorics*, pp. 57–95. Springer.

Kruskal, J. B. 1960. Well-quasi-ordering, the Tree Theorem, and Vazsonyi's conjecture. *Trans. Amer. Math. Soc.*, 95(2):210–225.

Marcone, A. 2001. Fine analysis of the quasi-orderings on the power set. *Order*, 18(4):339–347.

Rado, R. 1954. Partial well-ordering of sets of vectors. *Mathematika*, 1(2):89–95.

Robertson, N. and Seymour, P. D. 2004. Graph minors. XX. Wagner's conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325–357.

Schmidt, D. 1981. The relation between the height of a well-founded partial ordering and the order types of its chains and antichains. *Journal of Combinatorial Theory, Series B*, 31(2):183–189.

Schmidt, D. 2020. Well-partial orderings and their maximal order types. In **Schuster, P., Seisenberger, M., and Weiermann, A.**, editors, *Well Quasi-Orders in Computation, Logic, Language and Reasoning*, volume 53 of *Trends in Logic*, chapter 12, pp. 351–391. Springer.

Schmitz, S. 2019. The parametric complexity of lossy counter machines. In *Proc. 46th Int. Coll. Automata, Languages, and Programming (ICALP 2019), Patras, Greece, July 2019*, volume 132 of *Leibniz International Proceedings in Informatics*, pp. 129:1–129:15. Leibniz-Zentrum für Informatik.

Van der Meeren, J., Rathjen, M., and Weiermann, A. 2015. Well-partial-orderings and the big Veblen number. *Archive for Mathematical Logic*, 54(1–2):193–230.

Vialard, I. 2023. Ordinal measures of the set of finite multisets. In *Proc. 48th Int. Symp. Math. Found. Comp. Sci. (MFCS 2023), Bordeaux, France, Aug. 2023*, volume 272 of *Leibniz International Proceedings in Informatics*, pp. 87:1–87:15.

Leibniz-Zentrum für Informatik.

Vialard, I. 2024a. *Measuring well-quasi orders and complexity of verification.* PhD thesis, Univ. Paris-Saclay.

Vialard, I. 2024b. On the width of the cartesian product of ordinals. *Order*, 42(1):37–58.

Weiermann, A. 2009. A computation of the maximal order type of the term ordering on finite multisets. In *Proc. 5th Conf. Computability in Europe (CiE 2009), Heidelberg, Germany, July 2009*, volume 5635 of *Lecture Notes in Computer Science*, pp. 488–498. Springer.

Wolk, E. S. 1967. Partially well ordered sets and partial ordinals. *Fund. Math.*, 60(2):175–186.

Appendix A. Appendix to Section 3 "Upper and lower bounds"

We provide the proofs missing in Section 3.

A.1 Structural lemmas

Lemma 3.2 (Sandwich Lemma). *Let $(A, \leq)$ be a wqo such that $\mathbf{o}(A) = \alpha + \beta$ for some ordinals α, β. Then there exists a partition $A = A_\alpha \uplus A_\beta$ of A with $\mathbf{o}(A_\alpha) = \alpha$, $\mathbf{o}(A_\beta) = \beta$, and such that $A_\alpha \sqcup A_\beta \leq_{\mathsf{aug}} A \leq_{\mathsf{aug}} A_\alpha + A_\beta$.*

Lemma 3.2 generalizes Thm. 3.2 from de Jongh and Parikh (1977) where the case $\beta = 1$ is handled.

Proof. Let A be a wqo such that $\mathbf{o}(A) = \alpha + \beta$. Then, by Theorem 2.13 of de Jongh and Parikh (1977), there exists a surjective reflection $f : \alpha + \beta \to A$. We let $A_\alpha = f(\{\gamma \mid \gamma < \alpha\})$ and $A_\beta = f(\{\gamma \mid \alpha \leq \gamma < \alpha + \beta\})$. We further assume that f makes $\mathbf{o}(A_\alpha)$ minimal: since the collection of all reflections from $\alpha + \beta$ to A is a set, the collection of all possible $\mathbf{o}(A_\alpha)$ is a set of ordinals, which admits a minimal element.

Now, since f is a reflection, for all $a \in A_\alpha, b \in A_\beta$, $a \not\geq b$. It follows that

$$A_\alpha \sqcup A_\beta \leq_{\mathsf{aug}} A \leq_{\mathsf{aug}} A_\alpha + A_\beta$$

which implies: $\mathbf{o}(A_\alpha) + \mathbf{o}(A_\beta) \leq \alpha + \beta \leq \mathbf{o}(A_\alpha) \oplus \mathbf{o}(A_\beta)$. Besides, f gives a reflection from α to A_α, and one from $\{\gamma \mid \alpha \leq \gamma < \alpha + \beta\}$ (which is isomorphic to β by left-cancellativity) to A_β, hence $\mathbf{o}(A_\alpha) \geq \alpha$ and $\mathbf{o}(A_\beta) \geq \beta$. But then $\alpha + \beta \geq \mathbf{o}(A_\alpha) + \mathbf{o}(A_\beta) \geq \alpha + \mathbf{o}(A_\beta)$ and therefore $\beta \geq \mathbf{o}(A_\beta)$ since ordinal addition is left-cancellative. Besides, since $\mathbf{o}(A_\alpha) \geq \alpha$, we can write $\mathbf{o}(A_\alpha) = \alpha + \gamma$ for some γ that must satisfy $\gamma + \beta = \beta$.

Then there exists a reflection $g : \alpha + \gamma \to A_\alpha$. We consider the reflection $f' : \alpha + \gamma + \beta \to A$ obtained by concatenating g and f restricted to $\{\gamma \mid \alpha \leq \gamma < \alpha + \beta\}$. Let $B_\alpha \overset{\text{def}}{=} f'(\{\delta \mid \delta < \alpha\})$. By construction $\mathbf{o}(B_\alpha) \geq \omega$. However, if $\gamma > 0$, then B_α is a $\not\geq$-residual of A_α, hence $\mathbf{o}(B_\alpha) < \mathbf{o}(A_\alpha)$. It follows that $\mathbf{o}(A_\alpha)$ is not minimal, which is absurd. Therefore, $\mathbf{o}(A_\alpha) = \alpha$. $\qquad\square$

A.2 Lower bounds on the width of a product

Definition A.1 (Džamonja et al. (2020)). A wqo A is *transferable* if for every $x_1, \ldots, x_n \in A$, $\mathbf{w}(A_{\not\geq\{x_1,\ldots,x_n\}}) = \mathbf{w}(A)$.

Lemma A.2. *For any wpo A, if $\mathbf{h}(A) < \omega$ then A is finite.*

Proof. Let $\mathbf{h}(A) = k < \omega$. Then one can sort every element of A into k finite antichains, depending on their rank in $\mathrm{Dec}(A)$. Since every antichain of a wqo is finite, A is finite. $\qquad\square$

Proposition A.3. *If* $\mathbf{w}(A) \geq \omega$ *is additively indecomposable, then* A *contains a substructure of width* $\mathbf{w}(A)$ *which is transferable.*

Proof. In this proof we consider A as a wpo (by quotienting by $\equiv$ if needed). Note that for any partition $A = A_1 \uplus A_2$, $\mathbf{w}(A) \leq \mathbf{w}(A_1) \oplus \mathbf{w}(A_2)$. Thus if $\mathbf{w}(A) \geq \omega$ is additively indecomposable and A_2 is finite, necessarily $\mathbf{w}(A_1) = \mathbf{w}(A)$. In other words, one can remove any finite number of elements from A without changing its width.

Since $\mathbf{w}(A) \geq \omega$, A is infinite, hence $\mathbf{h}(A) \geq \omega$ according to Lemma A.2.

Observe that $A_{\not\leq \{x_1,\ldots,x_n\}} = A \setminus \downarrow\{x_1, \ldots, x_n\}$ for any $x_1, \ldots, x_n \in A$.

We now proceed to prove the proposition by induction on $\mathbf{h}(A)$.

If $\mathbf{h}(A) = \omega$ then for every $x_1, \ldots, x_n \in A$, for $A_1 = \downarrow\{x_1, \ldots, x_n\}$, we have $\mathbf{h}(A_1) < \mathbf{h}(A) = \omega$ hence A_1 is finite according to Lemma A.2, and therefore of finite width. It follows that $A_2 \overset{\text{def}}{=} A \setminus A_1$ is of width $\mathbf{w}(A)$ since the latter is additively indecomposable and $\mathbf{w}(A) \leq \mathbf{w}(A_1) \oplus \mathbf{w}(A_2)$. Thus A is transferable.

If $\mathbf{h}(A) > \omega$, either A is transferable, in which case the proposition is proved, or it is not.

If A is not transferable then there exists $x_1, \ldots, x_n \in A$ such that $\mathbf{w}(A_2) < \mathbf{w}(A)$ where $A_2 \overset{\text{def}}{=} A \setminus A_1$ and $A_1 \overset{\text{def}}{=} \downarrow\{x_1, \ldots, x_n\}$. Since $\mathbf{w}(A)$ is additively indecomposable and $\mathbf{w}(A) \leq \mathbf{w}(A_1) \oplus \mathbf{w}(A_2)$, it follows that $\mathbf{w}(A_1) = \mathbf{w}(A)$. Moreover, since $\mathbf{w}(A) \geq \omega$ is additive indecomposable, then $\mathbf{w}(A_1') = \mathbf{w}(A)$ for $A_1' \overset{\text{def}}{=} A_{<x_1,\ldots,x_n}$. And $\mathbf{h}(A_1') < \mathbf{h}(A)$ by the 2descent equation, while $\mathbf{h}(A_1') \geq \omega$ because $\mathbf{w}(A_1') \geq \omega$. Therefore by induction A_1' has a transferable substructure of width $\mathbf{w}(A_1') = \mathbf{w}(A)$. $\qquad\square$

Recall the following property of transferable wqos:

Lemma A.4 (Džamonja et al. (2020, Thm. 4.16))**.** *If δ is an ordinal and B a transferable* wqo*, then* $\mathbf{w}(\delta \times B) \geq \mathbf{w}(B) \cdot \delta$.

This lemma can be generalized to any wqo A such that $\mathbf{o}(A) = \delta$.

Lemma A.5. *Suppose that B is a transferable* wqo *and A any* wqo*. Then* $\mathbf{w}(A \times B) \geq \mathbf{w}(B) \cdot \mathbf{o}(A)$.

Proof. Since $A \leq_{\text{aug}} \mathbf{o}(A)$, then $\mathbf{w}(A \times B) \geq \mathbf{w}(\mathbf{o}(A) \times B)$ and now Lemma A.4 applies. $\qquad\square$

Lemma 3.10 (Lower bound on the width of the Cartesian product)**.** *Let A, B be two* wqos *such that $\mathbf{w}(B) \geq \omega$ is additively indecomposable. Then* $\mathbf{w}(A \times B) \geq \mathbf{w}(B) \cdot \mathbf{o}(A)$.

Proof. Since $\mathbf{w}(B)$ is indecomposable, Proposition A.3 shows that it has a substructure C which is transferable and has width $\mathbf{w}(B)$.

Hence $A \times B \geq_{\text{st}} A \times C$ and $\mathbf{w}(A \times B) \geq \mathbf{w}(A \times C) \geq \mathbf{w}(B) \cdot \mathbf{o}(A)$ with Lemma A.5. $\qquad\square$

Appendix B. Appendix to Section 4 "Tightness of the bounds"

B.1 Proofs for Section 4.1 studying the family $\mathscr{H}_\alpha$

Remark B.1. $\alpha \odot \beta$ (introduced in Figure 3) is the *Jacobsthal multiplication*. It is a transfinite iteration of the natural sum, in the same way the ordinal product is a transfinite iteration of the ordinal sum, thus it is halfway between the ordinal and natural multiplication: $\alpha \cdot \beta \leq \alpha \odot \beta \leq \alpha \otimes \beta$ (Proposition 5.1.2 of Altman (2017)) for any ordinal α, β.

In the particular case where α is additive indecomposable ordinal, we have the equality $\alpha \cdot \beta = \alpha \odot \beta$, since $\alpha \oplus \alpha = \alpha + \alpha$.

978 **Lemma B.2.** *For every ordinal $\alpha > 0$,*

$$\sum_{\beta < \alpha} \omega^\beta = \begin{cases} \omega^{\alpha-1} \cdot 2 & \text{when } \alpha = \gamma + 1, \gamma \text{ limit} \\ \omega^{\alpha-1} & \text{otherwise} \end{cases}$$

979 *Proof.* We prove the result by induction on $\alpha > 0$.

980 For $\alpha = 1$ we have $1 = \omega^{\alpha-1}$.

981 For $\alpha = \gamma + 1$ with γ a limit ordinal, we have:

$$\sum_{\beta < \alpha} \omega^\beta = \left(\sum_{\beta < \gamma} \omega^\beta \right) + \omega^\gamma = \omega^\gamma + \omega^\gamma = \omega^{\alpha-1} \cdot 2 \,.$$

982 For $\alpha = \gamma + 2$ we have:

$$\sum_{\beta < \alpha} \omega^\beta = \sum_{\beta < \gamma+1} \omega^\beta + \omega^{\gamma+1} = \begin{cases} \omega^\gamma \cdot 2 + \omega^{\gamma+1} & \text{when } \gamma \text{ limit} \\ \omega^\gamma + \omega^{\gamma+1} & \text{otherwise} \end{cases}$$
$$= \omega^{\gamma+1} = \omega^{\alpha-1} \,.$$

983 For α a limit ordinal, $\sum_{\beta < \alpha} \omega^\beta = \sup_{\gamma < \alpha} \left\{ \sum_{\beta < \gamma} \omega^\beta \right\}$. Let $\gamma < \alpha$, then $\gamma + 2 < \alpha$ and

984 $\sum_{\beta < \gamma} \omega^\beta \leq \sum_{\beta < \gamma+2} \omega^\beta$. By induction hypothesis, we conclude that $\sum_{\beta < \gamma} \omega^\beta \leq$

985 $\omega^{\gamma+1} < \omega^{\alpha-1} = \omega^\alpha$. As a consequence, $\sum_{\beta < \alpha} \omega^\beta \leq \omega^\alpha$. Conversely, for every $\gamma < \alpha$,

986 $\omega^\gamma \leq \sum_{\beta < \gamma+1} \omega^\beta$, and by definition of ordinal exponentiation, $\omega^\alpha \leq \sum_{\beta < \alpha} \omega^\beta$. □

987 We are now ready to prove the main lemma.

988 **Proposition 4.2.** *For all ordinal α:*

$$\mathbf{h}(\mathcal{H}_{\omega^\alpha}) = \mathbf{w}(\mathcal{H}_{\omega^\alpha}) = \mathbf{o}(\mathcal{H}_{\omega^\alpha}) = \omega^\alpha.$$

989 *Proof.* We prove the result by induction on α.

990 - It's immediate for $\mathcal{H}_{\omega^0}$.
991 - Given $x \in \mathcal{H}_{\omega^1}$, $(\mathcal{H}_{\omega^1})_{\not\geq x}$ is a finite set, hence $\mathbf{o}(\mathcal{H}_{\omega^1}) \leq \omega$ according to Eq. (Do). Since
992 $\mathcal{H}_{\omega^1}$ is an infinite sum indexed by ω, $\omega \leq \mathbf{h}(\mathcal{H}_{\omega^1})$. Besides, $\mathcal{H}_{\omega^1}$ contains arbitrary large
993 antichains and thus $\mathbf{w}(\mathcal{H}_{\omega^1}) \geq \omega$. It follows from Corollary 2.13 that $\mathbf{h}(\mathcal{H}_{\omega^1}) = \mathbf{w}(\mathcal{H}_{\omega^1}) =$
994 $\mathbf{o}(\mathcal{H}_{\omega^1}) = \omega^1$.
995 - If $\alpha > 0$, Table 2 and the induction hypothesis give:

$$\mathbf{h}(\mathcal{H}_{\omega^{\omega^\alpha}}) = \mathbf{o}(\mathcal{H}_{\omega^{\omega^\alpha}}) = \sum_{\beta < \omega^\alpha} \omega^\beta$$

$$\mathbf{w}(\mathcal{H}_{\omega^{\omega^\alpha}}) = \sup_{\beta < \omega^\alpha} \omega^\beta$$

996 The second expression clearly evaluates to ω^{ω^α} as desired. Since ω^α is a limit ordinal,
997 Lemma B.2 immediately gives $\mathbf{h}(\mathcal{H}_{\omega^{\omega^\alpha}}) = \mathbf{o}(\mathcal{H}_{\omega^{\omega^\alpha}}) = \omega^{\omega^\alpha}$.
998 - For the last case: if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ in CNF then the induction hypothesis and Table 2
999 yield:

$$\mathbf{h}(\mathcal{H}_{\omega^\alpha}) = \mathbf{o}(\mathcal{H}_{\omega^\alpha}) = \omega^{\omega^{\alpha_1}} \cdot \ldots \cdot \omega^{\omega^{\alpha_n}} = \omega^\alpha$$

For the width, ω^{α_i} is additively indecomposable for $i \in [1, n]$, hence

$$\mathbf{w}(\mathscr{H}_{\omega^\alpha}) = \omega^{\omega^{\alpha_1}} \odot \cdots \odot \omega^{\omega^{\alpha_n}} \quad \text{according to Table 2,}$$
$$= \omega^{\omega^{\alpha_1}} \cdot \cdots \cdot \omega^{\omega^{\alpha_n}} \quad \text{(Remark B.1)}.$$

$\square$