

LABELLED WELL QUASI ORDERED CLASSES OF BOUNDED LINEAR CLIQUE-WIDTH

Aliaume Lopez

UNIVERSITY OF WARSAW



MFCS'2025
Warsaw, Poland



ZYGMUNT
ZALESKI
STICHTING

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Presented by Maël Dumas

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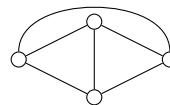
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Graphs and Induced Subgraphs



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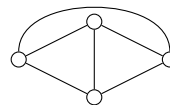
▷ Graphs are undirected, without self-loops



H

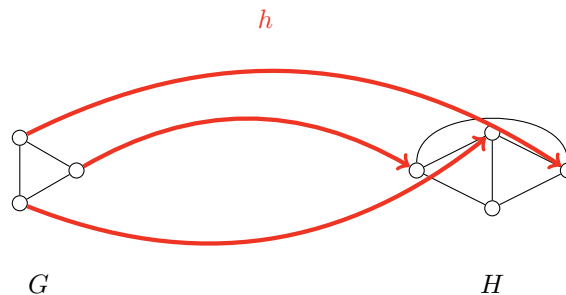
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 G  H

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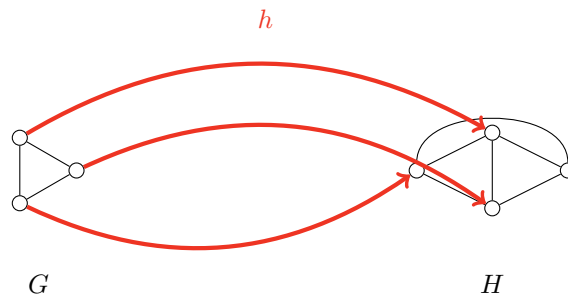
- ▷ Graphs are undirected, without self-loops
- ▷ Embeddings represent subsets of *vertices*



$$(x, y) \in E(G) \iff (h(x), h(y)) \in E(H)$$

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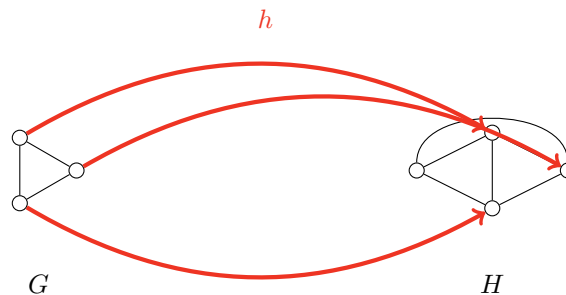
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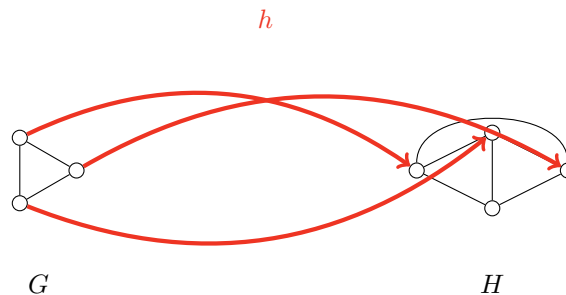
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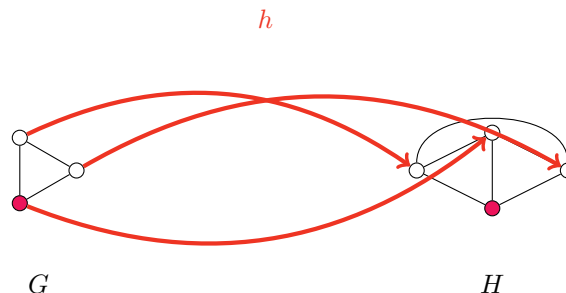
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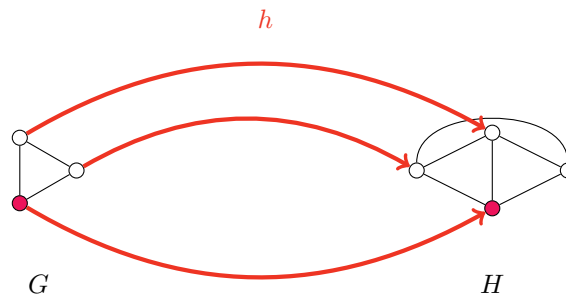
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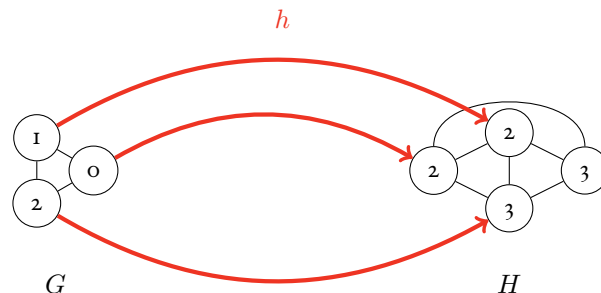
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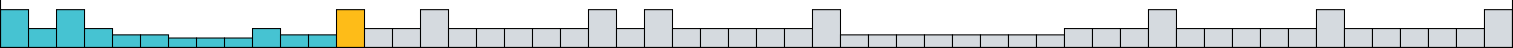


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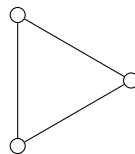
Freely Labeled Graphs

$\text{Label}_X(\mathcal{C})$ is the collection of all possible X -labellings of graphs in $\mathcal{C}$



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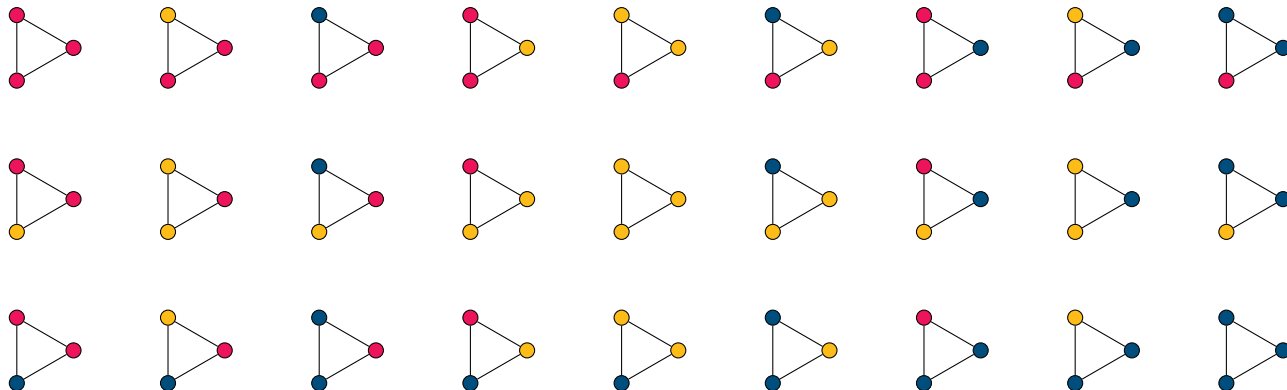
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$\mathcal{C} = \{C_3\}$ and $X = \{\cdot, \cdot, \cdot\}$



Well Quasi Ordered Clases of Graphs



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▷ A sequence of graphs $G_1, G_2, \dots$ is a **good sequence** if there exists a pair $i < j$ such that G_i *embeds* in G_j .

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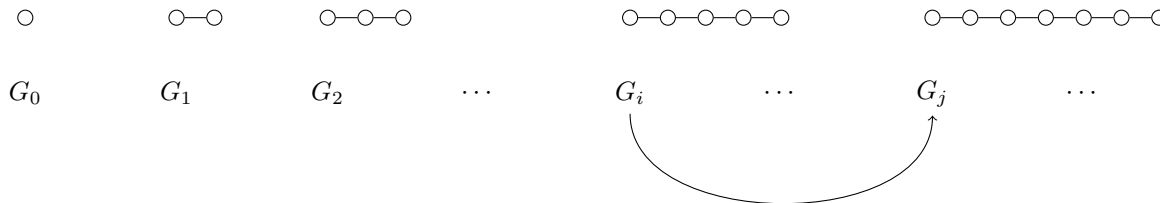
Bad Sequence

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Bad Sequence



Examples.

Cycles (NO), Paths (YES), Colored paths (NO)

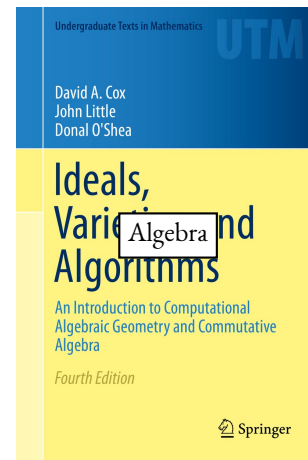
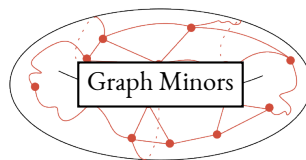
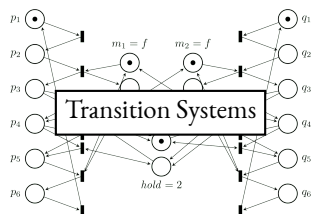
Examples.

Cliques : $\forall X$ wqo, X -wqo,
Paths : wqo, but not 2-wqo, Cycles : not wqo

Why care about Well-Quasi-Ordered Classes?



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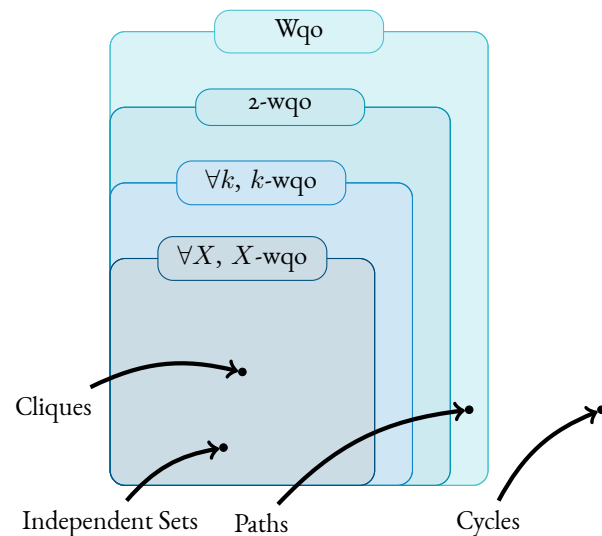


Related Work



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Inclusions of Properties



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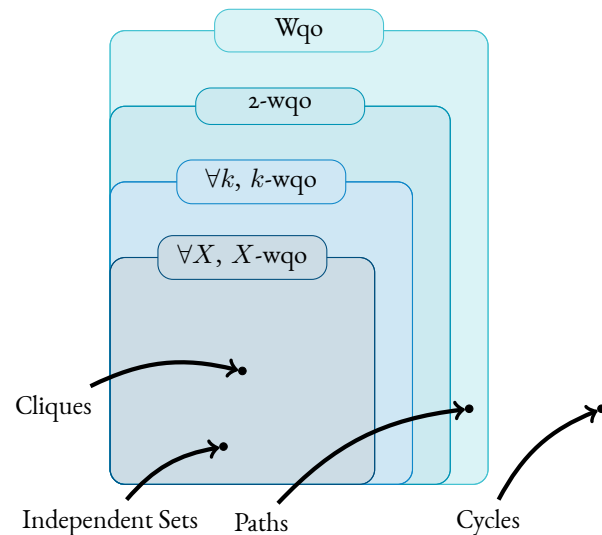
Conjecture 1 [Pouzet'72].

$(2\text{-wqo}) \implies (\forall k \in \mathbb{N}, k\text{-wqo})$

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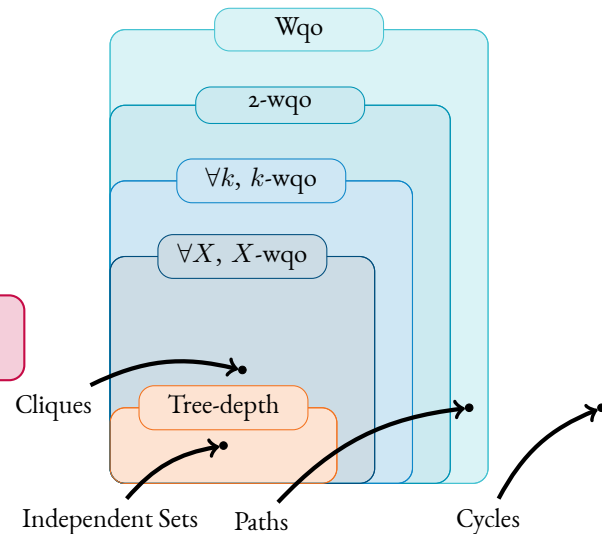
$(\forall k \in \mathbb{N}, k\text{-wqo}) \implies (\forall X \text{ wqo}, X\text{-wqo})$

Theorem [Ding'92].

Bounded tree-depth implies $\forall X, X\text{-wqo}$

the converse fails
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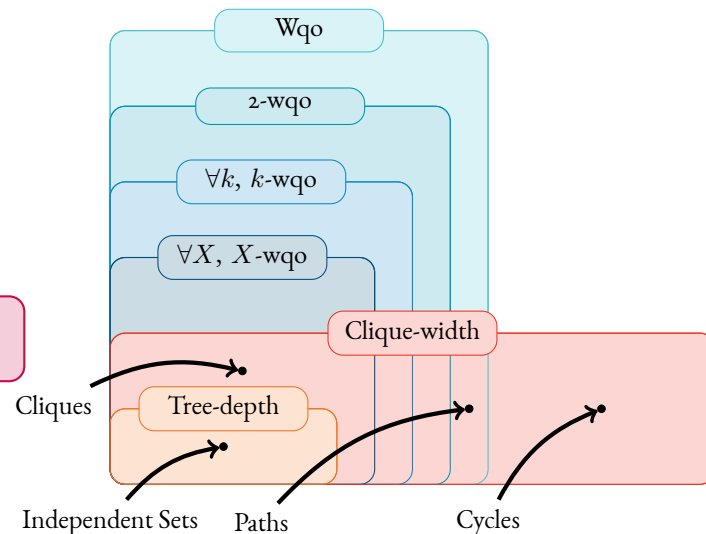
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For **some classes** of bounded clique-width, both conjectures hold

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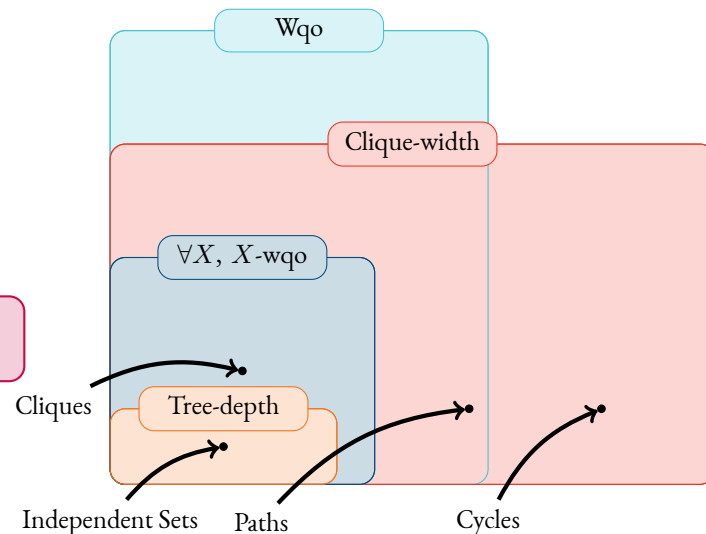
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Conjectured Inclusions of Properties



$\text{NLC}_Q^{\mathcal{F}}$ Expressions and Bounded Clique-Width

Relabel Expressions.

Select a finite set Q of colors, and a finite set $\mathcal{F}$ of functions from Q to Q .

- $\text{vertex}(q)$ for $q \in Q$,
- $\text{relabel}_f(g)$ for $f \in \mathcal{F}$, g an expression,
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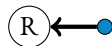


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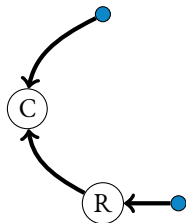


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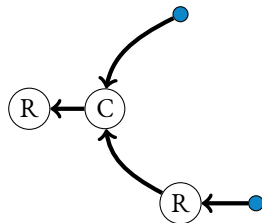


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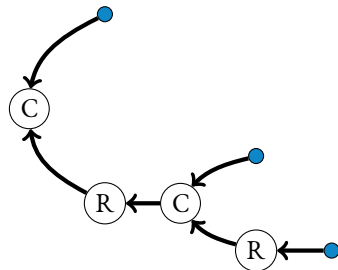


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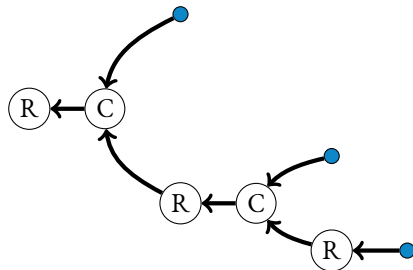


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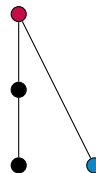
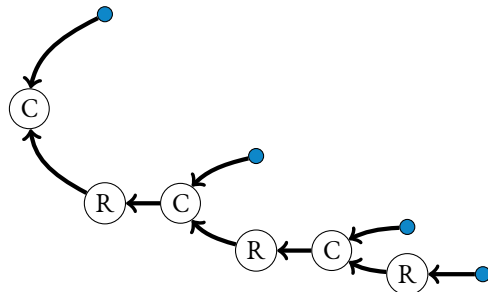


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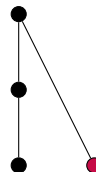
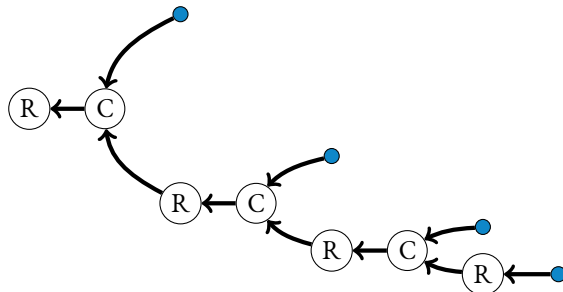


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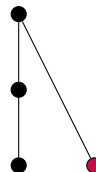
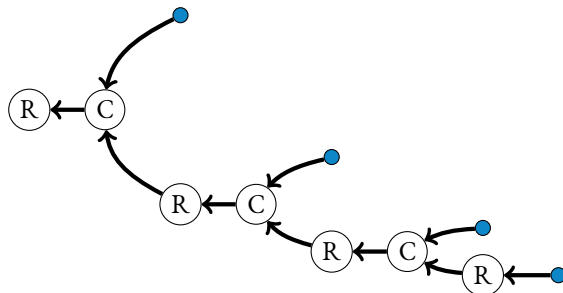
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Theorem [Dalgault, Rao, Thomassé'10].

For every $Q, \mathcal{F}$, one can decide whether $\text{NLC}_Q^{\mathcal{F}}$ is 2-wqo. Furthermore, the following are equivalent :

- $\text{NLC}_Q^{\mathcal{F}}$ is 2-wqo,
- $\text{NLC}_Q^{\mathcal{F}}$ is X -wqo, $\forall X$,
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Therefore, conjectures 1 and 2 are true for classes $\text{NLC}_Q^{\mathcal{F}}$

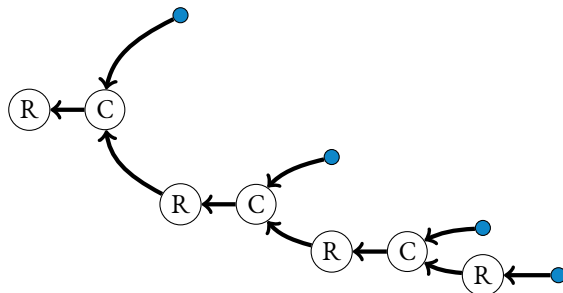


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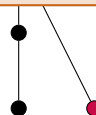
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A class has *bounded clique-width* if and only if it is **contained** in some $NLC_Q^{\mathcal{F}}$.

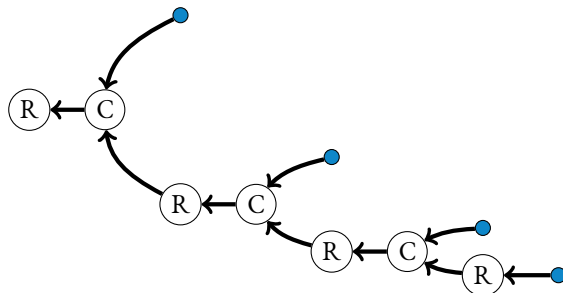


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Theorem [Courcelle].

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Problem.

This does not prove conjectures 1 and 2 for classes of bounded clique-width : subsets could still be WQO!

Results



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Setting.

We restrict ourselves to the **linear** NLC expressions, and we fix **how we add edges** once and for all :

$$\text{linNLC}_Q^{\mathcal{F}, P}$$

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linear shaped expression trees!

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Given $Q, \mathcal{F}, P$, the following properties are equivalent and **decidable** :

- $\text{linNLC}_Q^{\mathcal{F}, P}$ is X -wqo, for all X ,
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Theorem.

Given $Q, \mathcal{F}, P$, the following properties are equivalent and **decidable** :

- $\text{linNLC}_Q^{\mathcal{F}, P}$ is X -wqo, for all X ,
- $\text{linNLC}_Q^{\mathcal{F}, P}$ is k -wqo, for all k ,
- $\text{linNLC}_Q^{\mathcal{F}, P}$ is $(|\mathcal{F}|^3 \times 2)$ -wqo.

Corollary.

For all classes $\mathcal{C}$ of bounded **linear** clique-width, $\mathcal{C}$ is X -wqo (for all X) if and only if it is k -wqo (for all k).

Proof Sketch

Let $\mathcal{C} = \text{linNLC}_{\mathcal{Q}}^{\mathcal{F}, \mathcal{P}}$, be $(|\mathcal{F}|^3 \times 2)$ -wqo.

Prove that it is X -wqo for all X wqo.

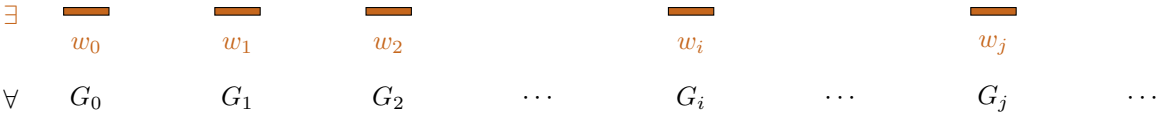
Proof Sketch

Let $\mathcal{C} = \text{linNLC}_{\mathcal{Q}}^{\mathcal{F}, \mathcal{P}}$, be $(|\mathcal{F}|^3 \times 2)$ -wqo.
Prove that it is X -wqo for all X wqo.

$\forall \quad G_0 \quad G_1 \quad G_2 \quad \dots \quad G_i \quad \dots \quad G_j \quad \dots$

Proof Sketch

Let $\mathcal{C} = \text{linNLC}_{\mathcal{Q}}^{\mathcal{F}, \mathcal{P}}$, be $(|\mathcal{F}|^3 \times 2)$ -wqo.
Prove that it is X -wqo for all X wqo.



By definition.
 $\forall G_i, \exists w_i$

Proof Sketch

Let $\mathcal{C} = \text{linNLC}_{\mathcal{Q}}^{\mathcal{F}, \mathcal{P}}$, be $(|\mathcal{F}|^3 \times 2)$ -wqo.

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Theorem [Simon].

$\exists d \in \mathbb{N}, \forall w_i, \exists t_i$
of depth at most d

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Theorem [Higman].

$\exists i < j, t_i \leq t_j$
words of words of words...

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Combinatorial Result

Theorem [Higman].

$\exists i < j, t_i \leq t_j$
words of words of words...

Conclusion(s)

Theorem.

Half of Pouzet's conjecture holds for classes of bounded linear clique-width :

$$\forall k \implies \forall X$$

Remark.

In their paper, Daligault, Rao, and Thomassé reproved part of Simon's factorisation theorem.

Motto.

Automata / Semigroup theory applied to well-quasi-orders and structural properties of graph classes

FUTURE WORK

Towards 2-wqo.

Better analysis of the proof should lead to 2-wqo

Towards trees.

Higman's lemma $\longrightarrow$ Kruskal's tree theorem

Simon's word factorisation $\longrightarrow$ Colcombet's tree factorization

Interpretations.

It is conjectured that for all hereditary classes $\mathcal{C}$:

$\mathcal{C}$ is 2-wqo



one cannot interpret all **paths** using existential first order formulas