Well-Quasi-Ordered Classes of Bounded Clique-Width

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HI!

Well-Quasi-Orders in $\ll 180$ seconds

Let $(Q, \leq)$ be a quasi-order, i.e. a reflexive and transitive relation on Q.

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Examples.

- $(\mathbb{N}, \leq)$
- $(\mathbb{N}^2, \leq)$ [Dickson]
- $(\Sigma^*, \leq^*)$ [Higman]
- $(\mathcal{G}, \leq_{\text{minor}})$ [Robertson-Seymour]
- $\dots$

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Non-Examples.

- $(\mathbb{Z}, \leq)$
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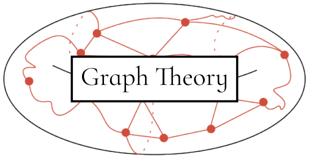

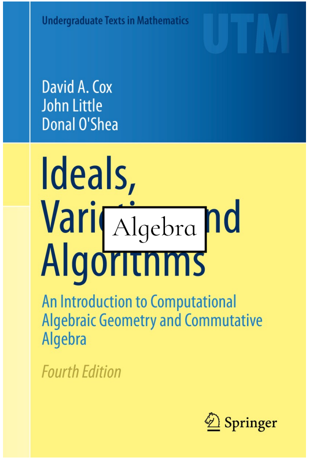

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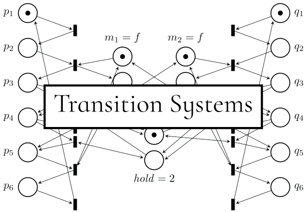

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Graph Theory : find new well-quasi-orders on all graphs
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Positive Results [Ding'92, folklore].
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Negative Result [folklore].
"every" quasi-order is representable as a class of graphs under the induced subgraph relation

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"Easier" Question.
Which classes of graphs can be used to build new well-quasi-orders ?

Higman's lemma $\rightsquigarrow$ labelling total orderings
Kruskal's theorem $\rightsquigarrow$ labelling trees
????????????????????? $\rightsquigarrow$ labelling graphs

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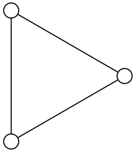

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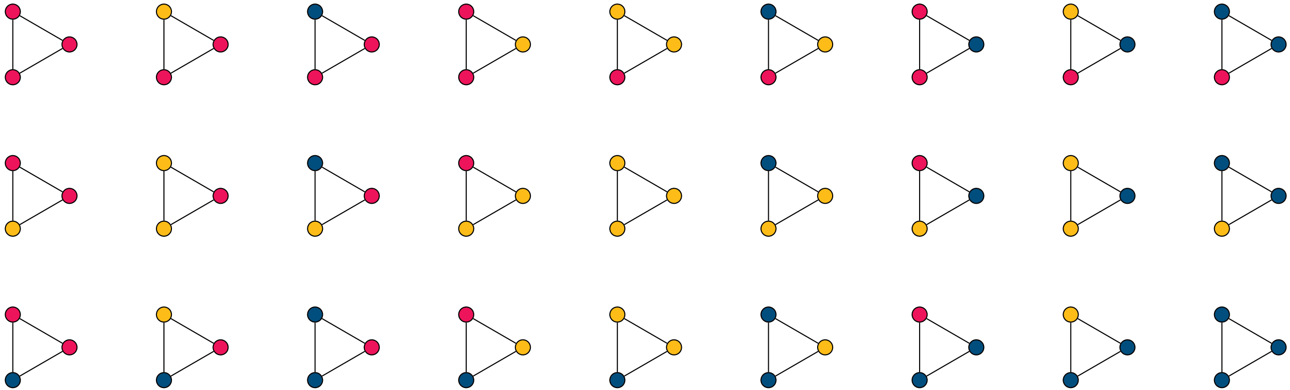
$\mathcal{C} = \{C_3\}$ and $X = \{\cdot, \cdot, \cdot\}$



Graph Theory

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Order Theory

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Graph Theory	Myself	Model Theory	Order Theory
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Graph Theory	Myself	First Conjecture	Model Theory	Second Conjecture	Order Theory
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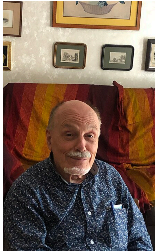

First Pouzet Conjecture : 2-WQO = ω-WQO

Second Pouzet Conjecture : ω-WQO = $\forall$-WQO

Schmitz's Conjecture : *finite paths* are the only obstruction to 2-WQO

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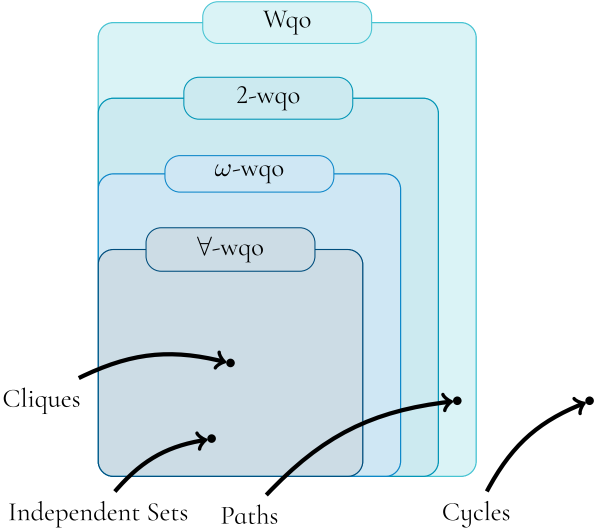

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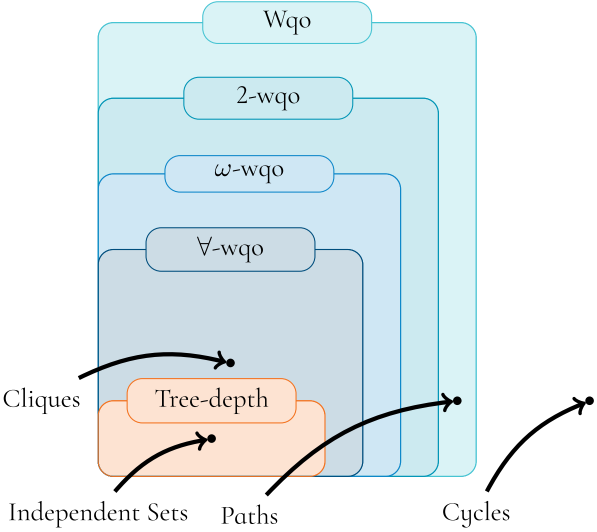
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Cliques
Independent Sets
Paths
Cycles

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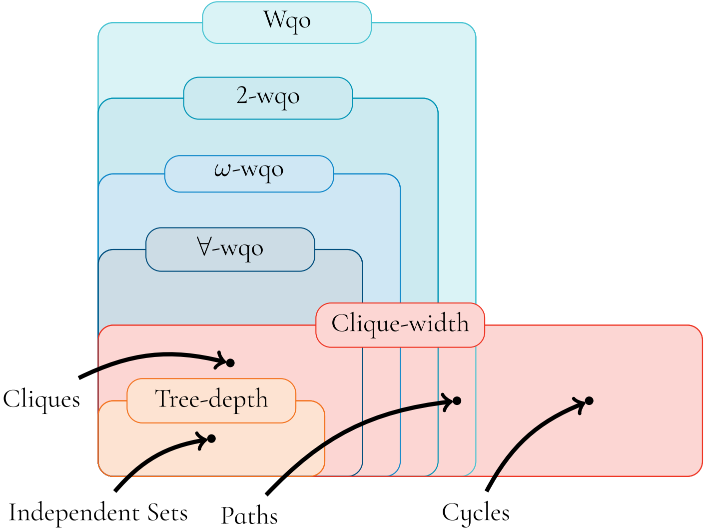
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\begin{aligned}
G = {} & \bullet_i && (i \in [k]) \\
\mid {} & G \oplus_f G && (f \subseteq [k] \times [k]) \\
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$\mathrm{NLC}^k_{\mathcal{F}}$

[DRT'10]

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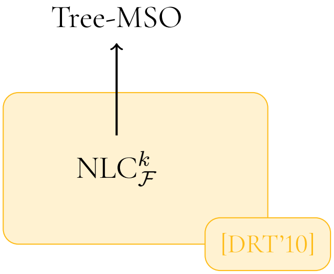

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$\mathcal{C} = \mathcal{I}(\mathsf{Tree}_\Sigma)$,
where $\mathcal{I}$ given by formulas $\phi_{\text{vertex}}(x), \phi_{\text{edge}}(x, y), \phi_{\text{domain}}$

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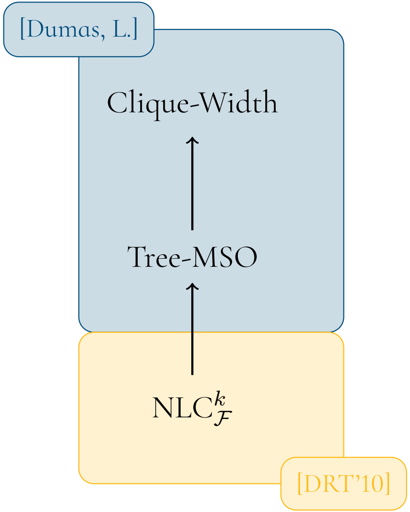

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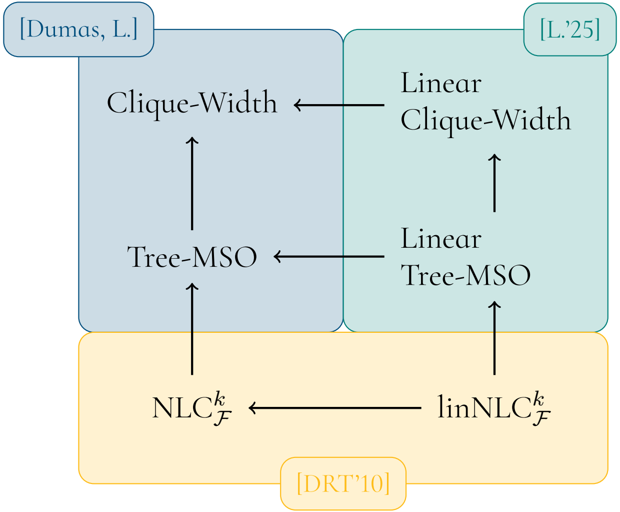

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Theorem 1. Assume hereditary and bounded clique-width.

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(as opposed to [DRT'10])

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Automata Theory Strikes Back

Towards Monoids.
Using *automata magic*, we can reduce the problem to talk about *finite monoids* $(M, \cdot)$ instead of *formulas* φ. E.g. $(\mathbb{Z}/2\mathbb{Z}, +)$.

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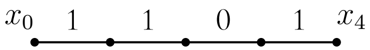
Trees now have **edges** labelled by M, and we can consider $[\![x : y]\!]$ the *product* of the labels along the path from x to y.



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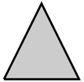
$\mathcal{I}:$  $\longmapsto \quad (\mathrm{leaves}(T), E)$

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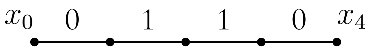
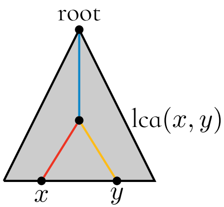


$$[\![x_0 : x_4]\!] = 0$$

How to Build Edges.
$E(x, y)$ is determined using the *values* of paths in the tree starting from x and y



Forward Ramseyan Splits

Recall.

We search an ordering on trees s.t.

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$$[\![x : y]\!] = [\![x : z_1]\!] \cdot [\![z_1 : z_3]\!] \cdot [\![z_3 : y]\!]$$

$$x \longrightarrow z_1 \longrightarrow z_2 \longrightarrow \cdots \longrightarrow z_3 \longrightarrow y$$

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More Automata Magic [Colcombet'07].
We can enrich the trees T with a *forward Ramseyan split* $\mathfrak{s}\colon T \to [n]$ allowing us to **short circuit** computations.

Attempt 0.
$T \leq_{\text{comp}} T'$ if there is a map $h\colon T \to T'$ preserving
- the root and leaves
- least common ancestors
- the *values* of paths in the tree.

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$$[\![x : y]\!] = [\![x : z_1]\!] \cdot [\![z_1 : z_3]\!] \cdot [\![z_3 : y]\!]$$

$$x \longrightarrow z_1 \longrightarrow z_2 \longrightarrow \cdots \longrightarrow z_3 \longrightarrow y$$

$\mathfrak{s}:$	$> k$	k	$> k$	k	$\geq k$	k	$> k$

"Fun" remark.
Requiring (B) to hold is *exactly* [DRT'10]...

Forward Ramseyan Splits

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We search an ordering on trees s.t.
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$\leq_{\mathrm{comp}}$

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Theorem [Dershowitz, Tzameret'03].
Trees with the *gap embedding* (respecting splits) are $\forall$-WQO!

$$\ominus \text{ relations} \xleftrightarrow{\quad \leq_{\text{gap}} \qquad\qquad\qquad \leq_{\text{comp}} \quad} \oplus \text{ relations}$$

$\leq_{\text{gap}}$: (A) ✗ (B) ✓

$\leq_{\text{comp}}$: (A) ✓ (B) ✗

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$\ominus$ relations $\longleftrightarrow$ $\oplus$ relations

$\leq_{\text{gap}}$	$\leq_{\text{super-nice}}$	$\leq_{\text{comp}}$
(A) ✗	(A) ✓?	(A) ✓
(B) ✓	(B) ✓?	(B) ✗

The Good, The Bad, and the Ugly Boughs

$$\llbracket x : y \rrbracket = \llbracket x : z_1 \rrbracket \cdot \llbracket z_1 : z_3 \rrbracket \cdot \llbracket z_3 : y \rrbracket$$

$$x \longrightarrow z_1 \longrightarrow z_2 \longrightarrow \cdots \longrightarrow z_3 \longrightarrow y$$

The Good, The Bad, and the Ugly Boughs

$$[\![x : y]\!] = [\![x : z_1]\!] \cdot [\![z_1 : z_3]\!] \cdot [\![z_3 : y]\!]$$

$$h : \begin{tikzcd} x \arrow[r] \arrow[d, bend right] & z_1 \arrow[r] \arrow[d, bend right] & z_2 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & z_3 \arrow[r] \arrow[d, bend left] & y \arrow[d, bend left] \\ h(x) \arrow[r] & h(z_1) \arrow[r] & h(z_2) \arrow[r] & \cdots \arrow[r] & h(z_3) \arrow[r] & h(y) \end{tikzcd}$$

GOAL : $[\![x : y]\!] = [\![h(x) : h(y)]\!]$

The Good, The Bad, and the Ugly Boughs

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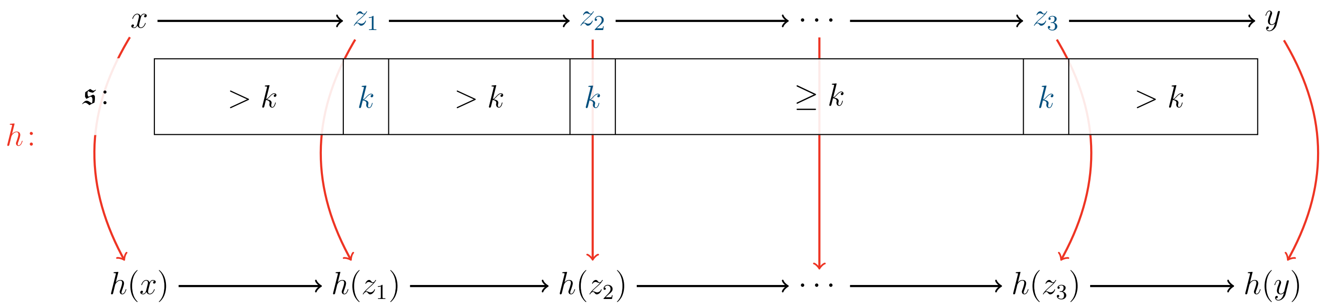

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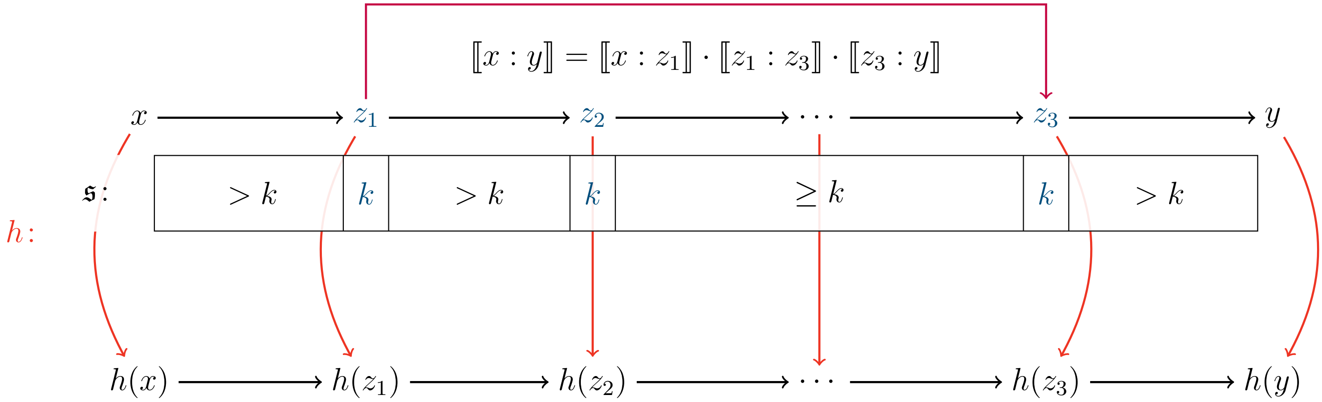

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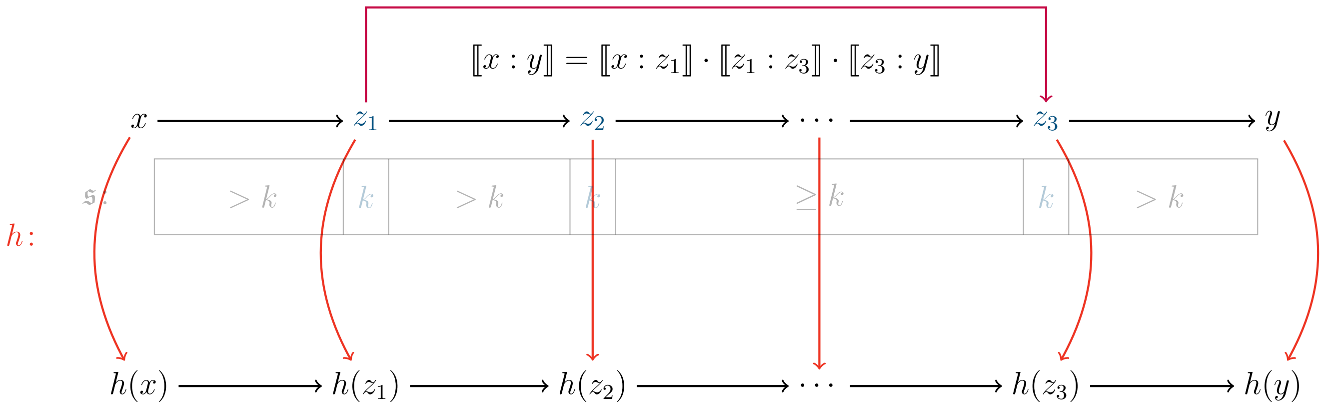

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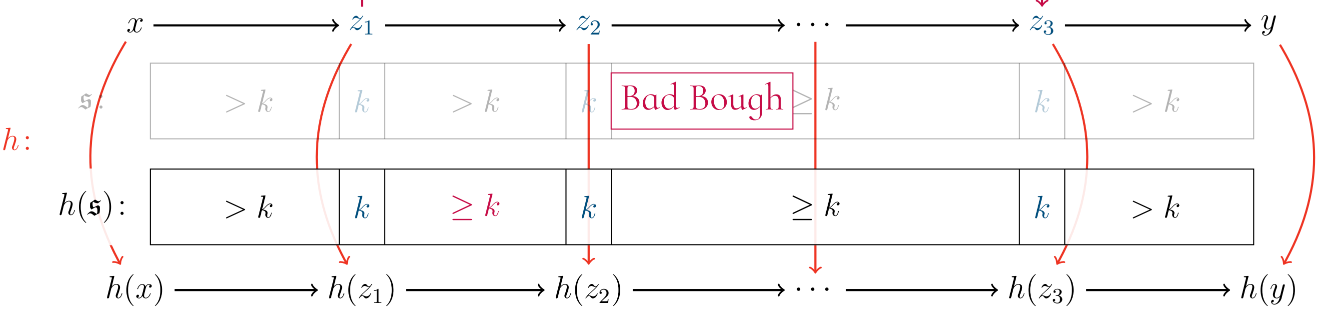

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The Good, The Bad, and the Ugly Boughs

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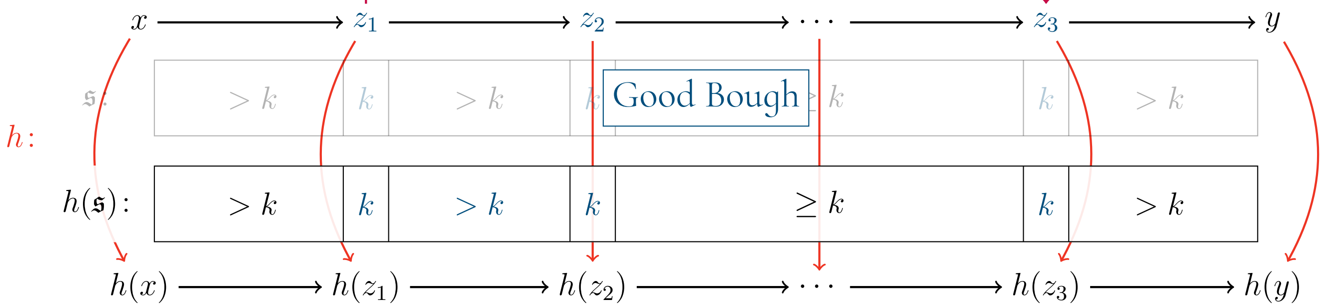

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The Good, The Bad, and the Ugly Boughs

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$x \longrightarrow y$

$\mathfrak{s} :$ k

$h :$

$h(\mathfrak{s}) :$ k

Key Technical Argument.
∞ bad boughs $\implies$ not 2-WQO
finitely bad boughs $\implies$ *order-preserving* ($\implies$ $\forall$-WQO)
(and it is decidable whether there are finitely many bad boughs)

$$h(x) \longrightarrow h(z_1) \longrightarrow h(z_2) \longrightarrow \cdots \longrightarrow h(z_3) \longrightarrow h(y)$$

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Conclusion

Theorem 1.
$\mathcal{C}$ *hereditary* + *bounded clique-width* :
1. $\mathcal{C}$ 2-WQO
2. $\mathcal{C}$ $\forall$-WQO

+ total ordering!

Theorem 2.
$\mathcal{C}$ Tree-MSO :
1. $\mathcal{C}$ is 2-WQO
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Theorem 3.
$\mathcal{C}$ *hereditary* + *bounded clique-width* :
1. $\mathcal{C}$ **not** 2-WQO
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Future Work.
1. All *hereditary* classes?
2. *interpret* paths?
3. Structural characterisation?
4. Beyond graphs?
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Future Work.
1. ~~All *hereditary* classes?~~

2-WQO $\Longrightarrow$ bounded clique width
[Duron, Mählmann, Toruńczyk]

2. *interpret* paths?
3. Structural characterisation?
4. Beyond graphs? ...