Polyregular Model Checking

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Bordeaux
M2F Seminar, 2025-09-30

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(Regular) Model Checking

Verify that a system satisfies a specification

(Regular) Model Checking

Verify that a system satisfies a specification

precondition $\{\varphi\}$ P $\{\psi\}$ postcondition

program

Hoare Triple

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Hoare Triple

Regular Model Checking :

- **Programs** : string-to-string programs
- **Specifications** : regular expressions

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HOARE TRIPLE

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Continuous functions :
$f^{-1}(L)$ is regular whenever L is a regular language

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Regular Model Checking of Continuous Functions

$$\varphi \implies P^{-1}(\psi)$$

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A zoo of transducer models

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Ariadne
Transducers

A zoo of transducer models

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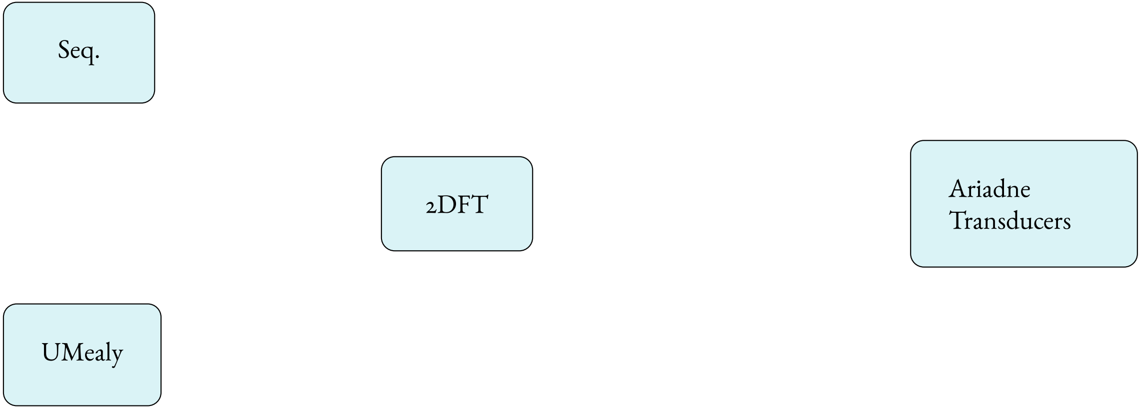

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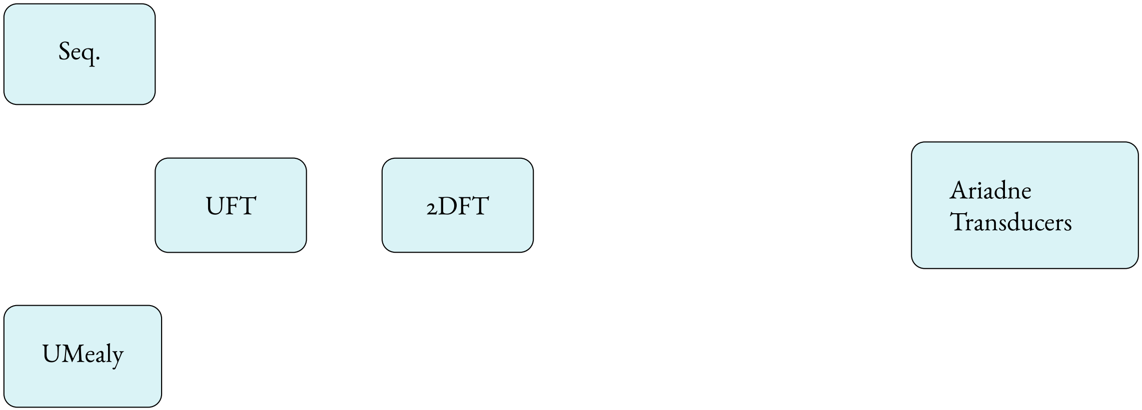

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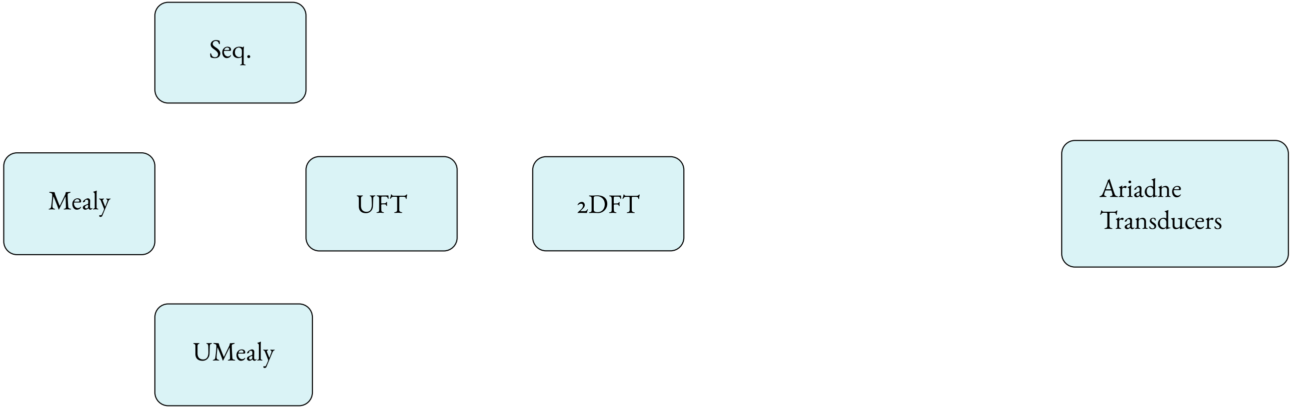

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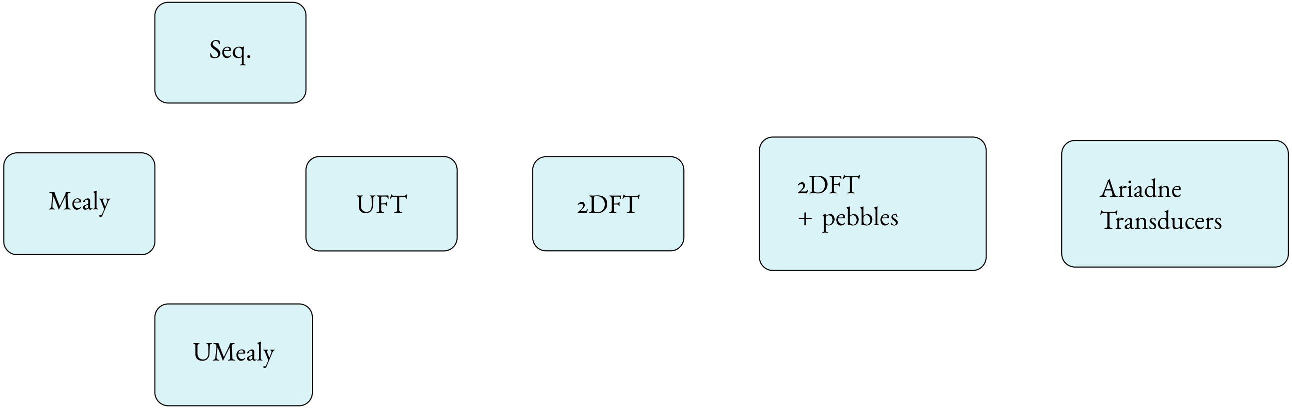

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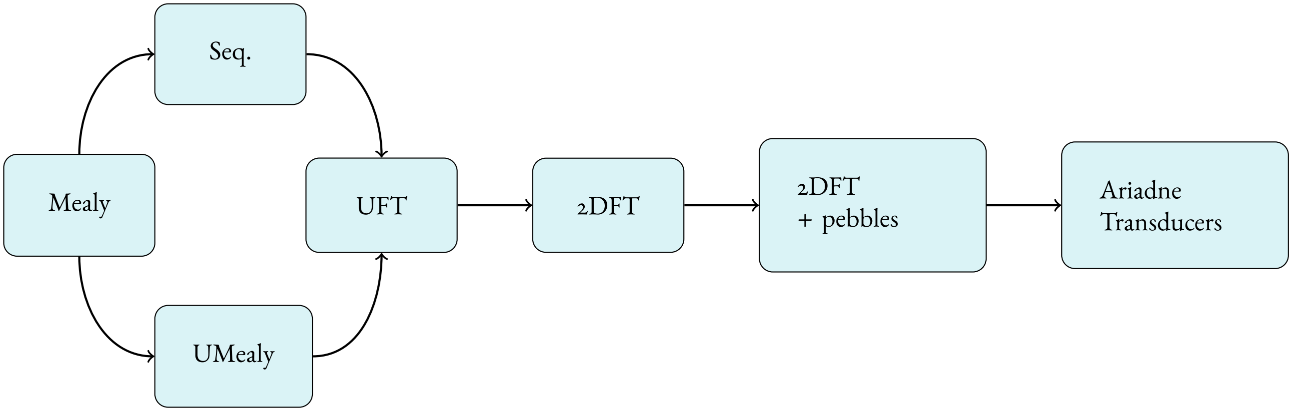

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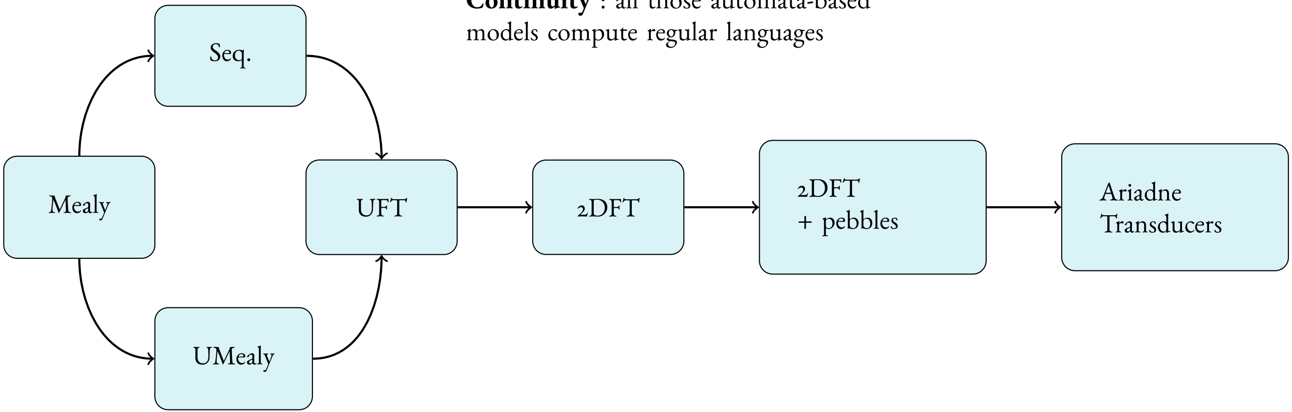

Continuity : all those automata-based models compute regular languages

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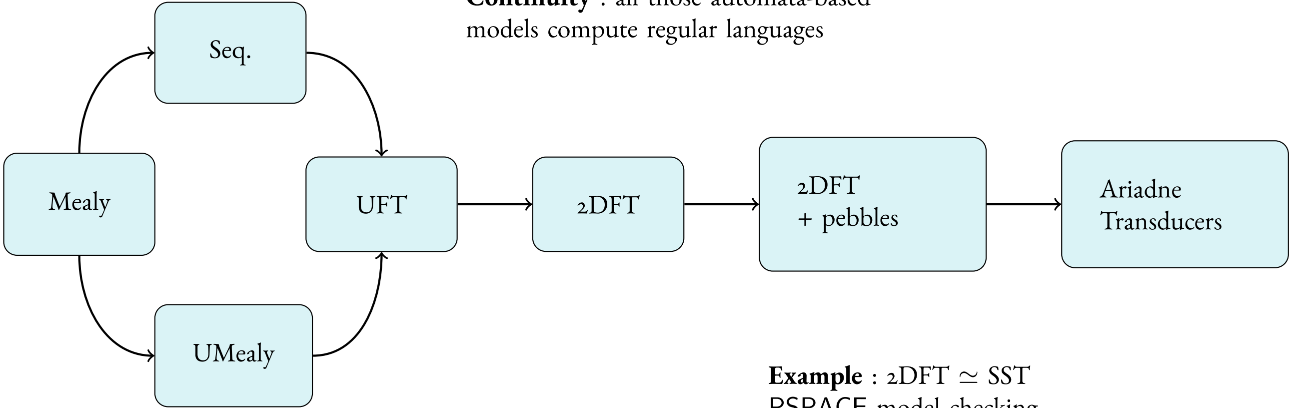

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Example : 2DFT $\simeq$ SST
PSPACE model checking
[Alur Černý, POPL'11]

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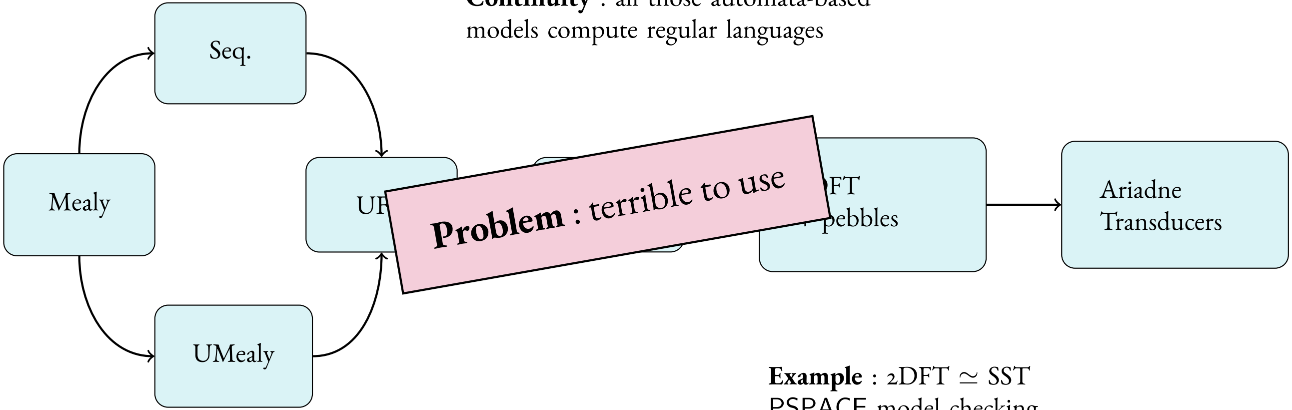

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Polyregular Functions

MSO interpretations

For-programs

Pebble transducers

List functions

Polyregular Functions

MSO
interpreta

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Pebble
transduce

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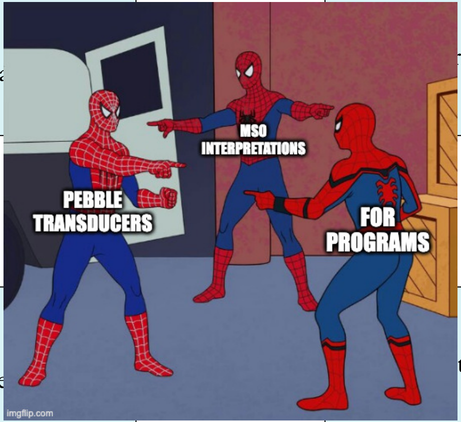

Let's design

```python
def getBetween(l, i, j):
    """ Get elements between i and j """
    for (k, c) in enumerate(l):
        if i <= k and k <= j: ①
            yield c ②

def containsAB(w):
    """ Contains "ab" as a subsequence """
    seen_a = False ③
    for (x, c) in enumerate(w):
        if c == "a": ④
                seen_a = True ⑤
        elif seen_a and c == "b":
            return True
    return False

def subwordsWithAB(word):
    """ Get subwords that contain "ab" """
    for (i,c) in enumerate(word): ⑥
        for (j,d) in reversed(enumerate(word)): ⑦
            s = getBetween(word, i, j) ⑧
            if containsAB(s):
                yield s
```

Fig. 1. A small Python program that outputs all subwords of a given word containing `ab` as a scattered subword

Let's design

Rules of the fight

- **lists** (2)
- **loops** (6) and (7)
- **variables** (6)
- **equality** (4)
- **tests** (1)
- **shadowing** no nay never
- **functions** no boolean inputs
- **updates** (3) and (5)

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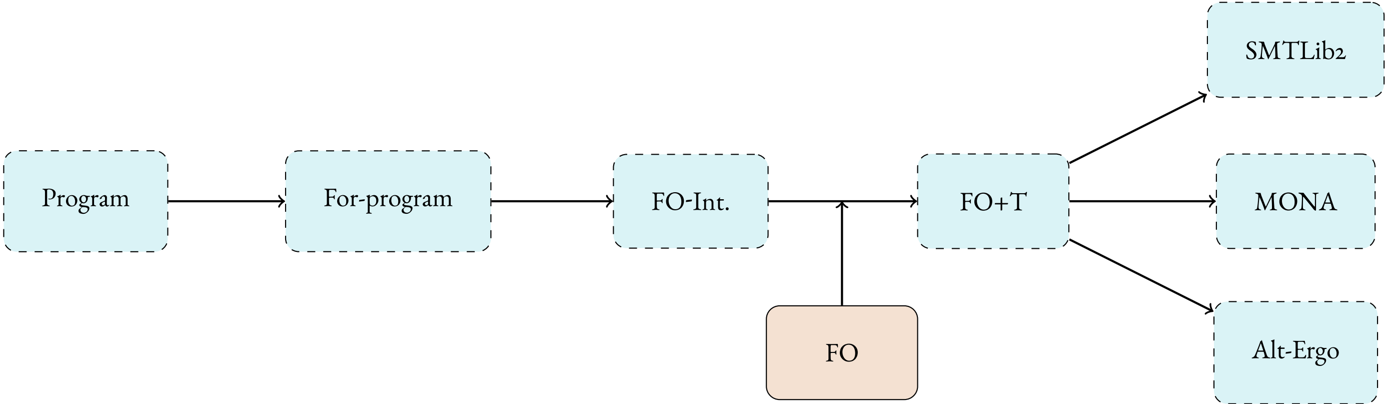
DEMO

Anatomy of a For(program checker)



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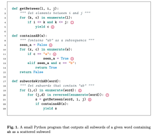

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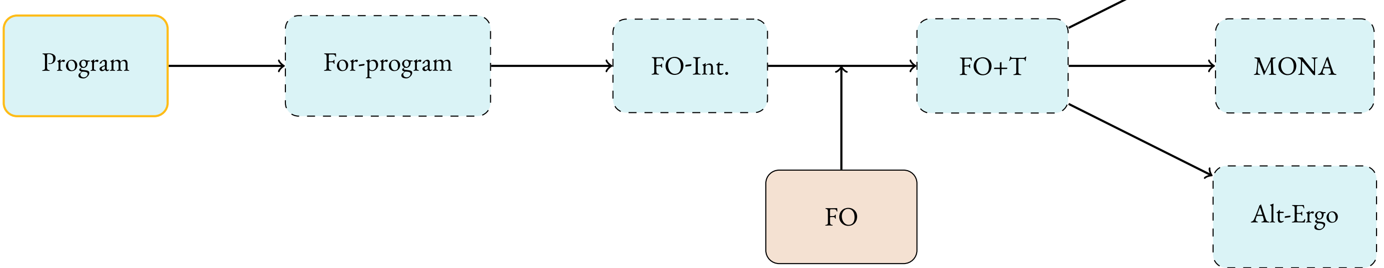

Simple for programs

Rules of the fight

- **functions** no no no
- **lists** no!
- **variables** only booleans / positions

```
seen_space_top = False ①
# first we handle all words except of the final one
for i in input: ②
    seen_space = False ③
    if label(i) == ' ': ④
        for j in reversed(input): ⑤
            if j < i:
                if label(j) == ' ':
                    seen_space = True
                if not seen_space:
                    print(label(j)) ⑥
        print(' ') ⑦

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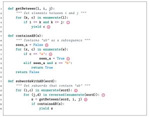

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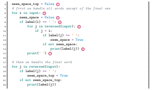

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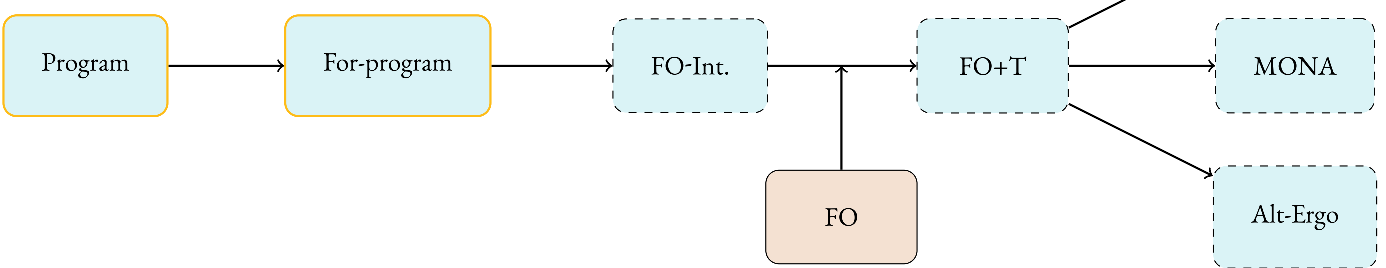

First-Order Interpretations

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Rules of the fight

tags finite set tags

arities $\mathsf{ar} \colon \mathsf{tags} \to \mathbb{N}$

domain first order formulas φ^t_{dom}

output letters $\mathsf{out} \colon \mathsf{tags} \to A + \mathbb{N}$

output order first order formulas $\varphi^{t_1,t_2}_{\leq}$

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$\varphi^{\texttt{printB}}_{\mathrm{dom}}(x) : x =_L \texttt{b} \quad \varphi^{\texttt{copy}}_{\mathrm{dom}}(x) : x \neq_L \texttt{b}$

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Fig. 4. The `swapAsToBs` interpretation.

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a u t o m a t e s
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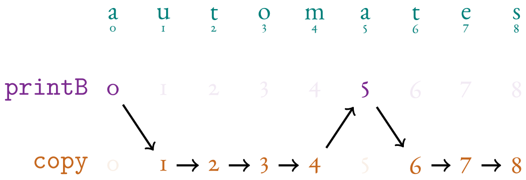

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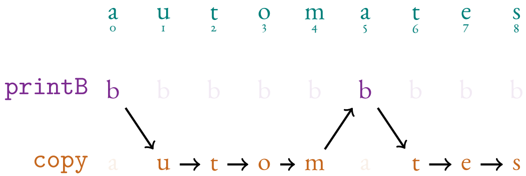

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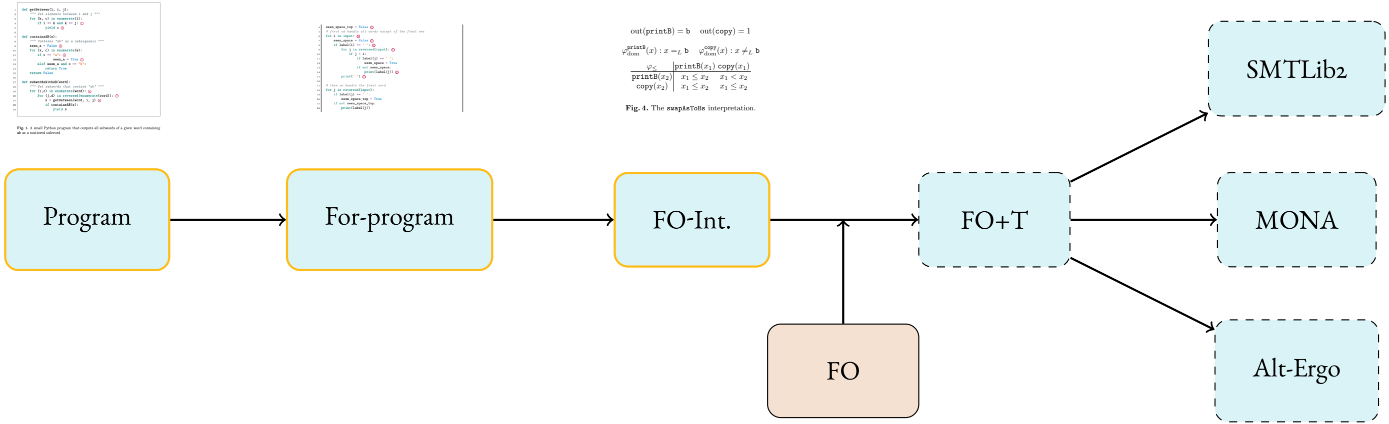

First-Order Logic… with tags!

$\varphi := \exists x : \mathsf{tag}, \varphi \mid \exists x : \mathsf{pos}, \varphi \mid \neg \varphi \mid \varphi_1 \wedge \varphi_2 \mid x =_L \mathsf{a} \mid x \leq y \mid x = \mathsf{t}$

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$$\mathsf{FO} \xrightarrow{f \in \mathsf{FO\text{-}I}} \mathsf{FO{+}T}$$

$$\forall w, f(w) \models \varphi \iff w \models f(\varphi)$$

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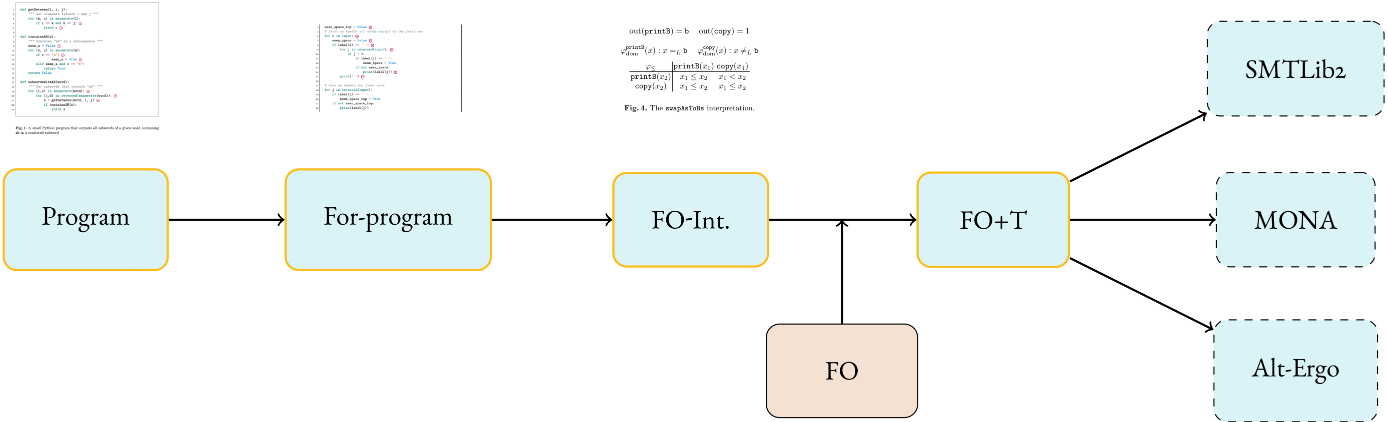
$$f(\forall_x \psi) := \forall_{t_x \in \mathsf{tags}} \forall_{x_1, \dots, x_{\mathsf{ar}(f)}} \left(\mathsf{dom}(t_x, x_1, \dots, x_{\mathsf{ar}(f)}) \Rightarrow f(\psi)\right)$$

$$\mathsf{dom}(t, x_1, \dots, x_{\mathsf{ar}(t)}) := \bigvee_{t' \in \mathsf{tags}} \left(t = t' \wedge \varphi^{t'}_{\mathsf{dom}}(x_1, \dots, x_{\mathsf{ar}(t')})\right)$$

$$f(x \leq y) := \bigvee_{t_1, t_2 \in \mathsf{tags}} \left(t_x = t_1 \wedge t_y = t_2 \wedge \varphi^{t_1, t_2}_{\leq}(x_1, \dots, x_{\mathsf{ar}(t_1)}, y_1, \dots, y_{\mathsf{ar}(t_2)})\right)$$

$$f(x =_L \mathsf{a}) := \left(\bigvee_{t \in \mathsf{tags} \wedge \mathsf{out}(t) = \mathsf{a}} t = t_x\right) \vee \left(\bigvee_{t \in \mathsf{tags} \wedge \mathsf{out}(t) \notin A} (t = t_x \wedge x_{\mathsf{out}(t)} =_L \mathsf{a})\right)$$

Anatomy of a For(program checker)

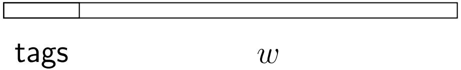


Calling solvers for help

Calling solvers for help

MONA

Solves : WS1S/WS2S over words



Calling solvers for help

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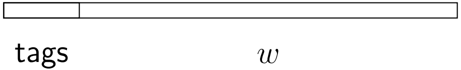

SMTLib2

Solves : First order theories

- `DT` : `tags`
- `UF` : $w : \mathbb{N} \to A + \bot$
- `LIA` : positions

CALLING SOLVERS FOR HELP

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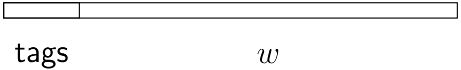

Complete but slow

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Incomplete but fast

Calling solvers for help

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tags w

Complete but sl

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omplete but fast

filename	FP size	FP l.d.	FP b.d.	S.FP size	S.FP l.d.	S.FP b.d.	FO-I size	FO-I q.r.
identity.pr	3	1	0	2	2	0	1	0
reverse.pr	3	1	0	2	2	0	1	0
subwords_ab.pr	24	2	1	15	4	3	956	14
map_reverse.pr	36	2	1	18	4	1	285	5
prefixes.pr	6	2	0	5	3	0	2	0
get_last_word.pr	18	1	1	23	4	2	8553	15
get_first_word.pr	22	1	1	5	2	0	103	4
compress_as.pr	12	1	1	12	3	2	209	10
litteral_test.pr	29	1	1	129	3	12	3.2×10^4	82
bibtex.pr	110	2	1	802	6	29	13.7×10^6	136

Calling solvers for help

MONA

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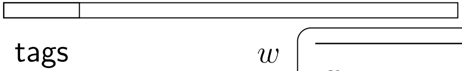

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FO model checking on words is TOWER-complete [Stockmeyer, 1974]

Anatomy of a For(program checker)

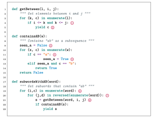
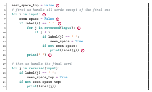


Fig. 1. A small Python program that outputs all subwords of a given word containing ab as a scattered subword



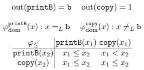
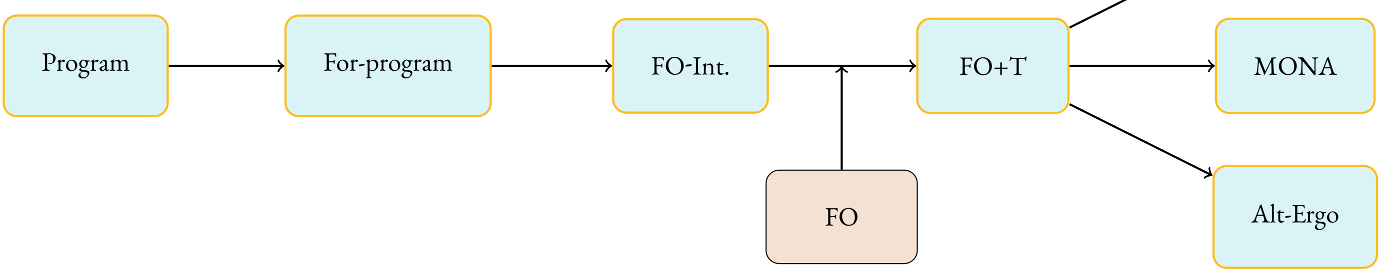


Fig. 4. The swapAsToBs interpretation.



Compiling to First Order

```python
seen_space_top = False ①
# first we handle all words except of the final one
for i in input: ②
    seen_space = False ③
    if label(i) == ' ': ④
        for j in reversed(input): ⑤
            if j < i:
                if label(j) == ' ':
                    seen_space = True
                if not seen_space:
                    print(label(j)) ⑥
        print(' ') ⑦

# then we handle the final word
for j in reversed(input):
    if label(j) == ' ':
        seen_space_top = True
    if not seen_space_top:
        print(label(j))
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                    print(label(j)) ⑥    t1
        print(' ') ⑦                     t2

# then we handle the final word
for j in reversed(input):
    if label(j) == ' ':
        seen_space_top = True
    if not seen_space_top:
        print(label(j))                  t3
```

Tags : tags $= \{t_1, t_2, t_3\}$

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```

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                    print(label(j)) ⑥        t1
        print(' ') ⑦                          t2

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Order : Lexicographic based on positions (QF)

Compiling to First Order

```
seen_space_top = False (1)
# first we handle all words except of the final one
for i in input: (2)
    seen_space = False (3)
    if label(i) == ' ': (4)
        for j in reversed(input): (5)
            if j < i:
                if label(j) == ' ':
                    seen_space = True
                if not seen_space:
                    print(label(j)) (6)      t1
        print(' ') (7)                       t2

# then we handle the final word
for j in reversed(input):
    if label(j) == ' ':
        seen_space_top = True
    if not seen_space_top:
        print(label(j))                      t3
```

Tags : tags $= \{t_1, t_2, t_3\}$

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Order : Lexicographic based on positions (QF)

Domain : ... difficult part!

values of the boolean variables?

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```

t_3

Tags : tags $= \{t_1, t_2, t_3\}$

…, ar$(t_3) = 1$

…space, out$(t_3) = j$

… positions (QF)

…riables ?

Program formulas

- FO + input (pos,bool) / output (bool)
- Can be composed easily
- Can implement if-then-else
- Can implement loops

One can write a program formula to compute the boolean variables at a given program position.

From High to Low

From High to Low

New operator : generator expressions.

- $\mathsf{gen}(s)$
- Can be used in place of a list / boolean
- Captures list variables but not boolean variables
- Simulate function calls

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Example code :

```
for (i,x) in enumerate(gen( expr )) ...
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Easy rewriting steps :

1. Remove literals
2. Remove functions
3. Remove boolean generators
4. Remove let expressions
5. Push boolean introductions upwards

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Easy rewriting steps :

1. Remove literals
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4. Remove let expressions
5. Push boolean introductions upwards

Print all but first, program "s"

```
b = False
for (i,x) in enumerate(u):
   if b:
      yield x
   else:
      b = True
```

What are the following programs doing?

```
for (i,x) in reverse(enumerate(s)):
   yield x
```

```
for (i,x) in enumerate(s):
   yield x
```

Forward Loop Elimination

```
for (i,x) in enumerate(s):
    for (j,y) in enumerate(s):
        if i == j:
            yield x
```

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Idea : substitute the body of the loop in s.
$s[\mathsf{yield} e \mapsto \ldots]$.

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Problem : We lost the index variable i!

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- i can only be tested against positions of s
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```
b1 = False
for (i1,x1) in enumerate(s):
   if b1:

   else:
      b1 = True
```

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b1 = False
for (i1,x1) in enumerate(s):
    if b1:
        b2 = False
        for (i2,x2) in enumerate(s):
            if b2:

            else:
                b2 = True
    else:
        b1 = True
```

Forward Loop Elimination

```
for (i,x) in enumerate(s):
    for (j,y) in enumerate(s):
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b1 = False
for (i1,x1) in enumerate(s):
    if b1:
        b2 = False
        for (i2,x2) in enumerate(s):
            if b2:
                if i1 = i2:
                    yield x1
            else:
                b2 = True
    else:
        b1 = True
```

Backward Loop Elimination

```
for (i,x) in reverse(enumerate(s)):
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Solution :

- Compute a *superset* of the reachable yields in the reversed order
- For every yield, check that it would be reachable
- If so, perform the rest of the computation

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Remark : *this is the proof that* polyregular functions *are closed under composition.*

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```
for (i,x) in reverse(enumerate(s)):
  yield x
```

Problem : reverse a non reversible computation !

Solution :

- Compute a *superset* of the reachable yields in the reversed order
- For every yield, check that it would be reachable
- If so, perform the rest of the computation

```
for (i1, x1) in reversed(enumerate(u)):
  for (i2, x2) in enumerate(u):
    b2 = False
    if b2:

    else:
      b2 = True
```

Remark : *this is the proof that* polyregular functions *are closed under composition.*

Backward Loop Elimination

```
for (i,x) in reverse(enumerate(s)):
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```

Problem : reverse a non reversible computation !

Solution :

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```
for (i1, x1) in reversed(enumerate(u)):
  for (i2, x2) in enumerate(u):
    b2 = False
    if b2:
      if i1 = i2:
        yield x1
    else:
      b2 = True
```

Remark : *this is the proof that* polyregular functions *are closed under composition.*

In the end...

CAV Artifact Evaluation ★ Available

Polyregular Model Checking

Aliaume Lopez[1] [0000-0002-4205-327X]⋆ and Rafał Stefański[1] [0000-0002-8439-4056]⋆⋆

University of Warsaw

CAV Artifact Evaluation ★★★ Reusable

And more :

- Haskell implementation + webapp
- Nix / Docker / reproducible builds
- Symbolic alphabets
- Some optimisations

Future work :

- Comparison with other models
- Better interface with solvers
- Composable checks
- Monadic second order logic