Well-Quasi-Orders

and

Logic on Graphs

ZYGMUNT
ZALESKI
STICHTING

Aliaume Lopez
University of Warsaw

Les Houches
FMT'25, 2025-05-29

https://www.irif.fr/~alopez/

Well-Quasi-Orders 101

Well-quasi-order (WQO) $(X, \leq)$

$$\begin{tikzcd} x_0 & x_1 & x_2 & \dots & x_i \arrow[rr, bend right, "\leq"'] \arrow[rr, phantom, "\exists"] & \dots & x_j & \dots \end{tikzcd}$$

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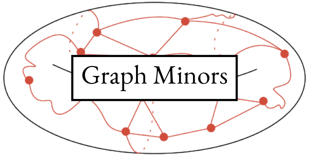
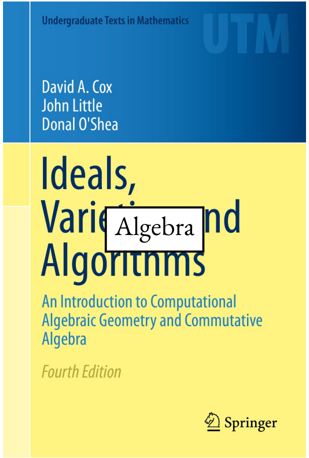
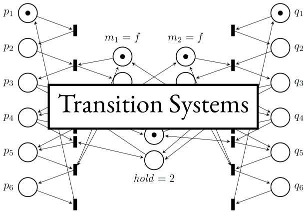
$$x_0 \qquad x_1 \qquad x_2 \qquad \ldots \qquad x_i \qquad \ldots \qquad x_j \qquad \ldots$$

$\exists$

$\leq$

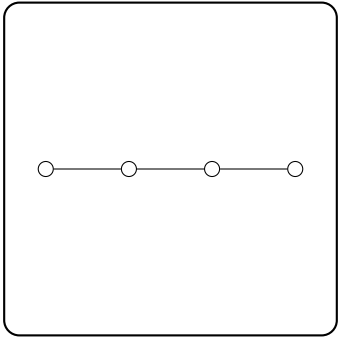






Is the following class WQO for induced subgraphs

YES NO



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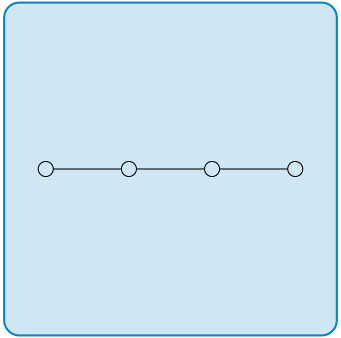

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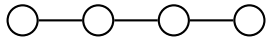

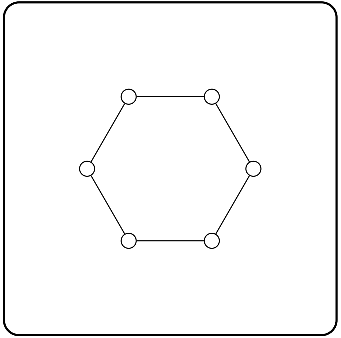

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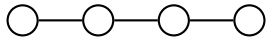

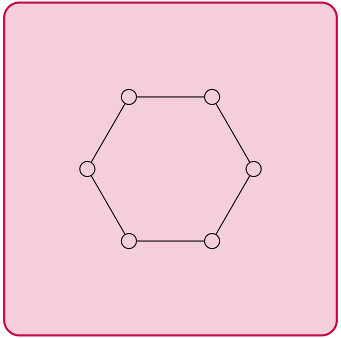

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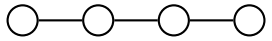

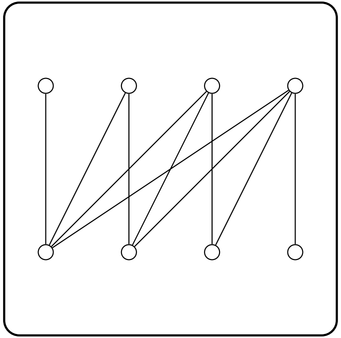

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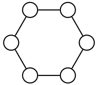

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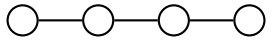

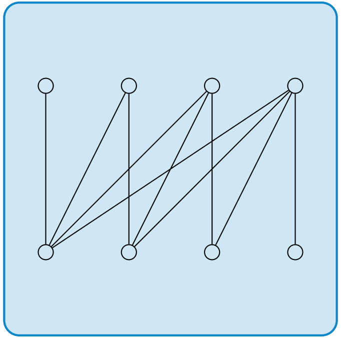

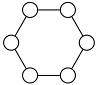

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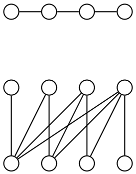

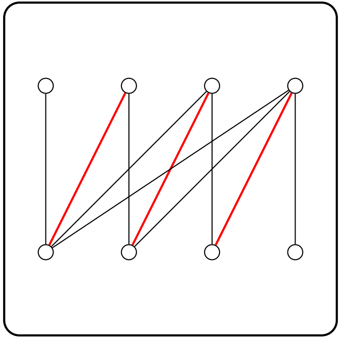

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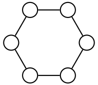

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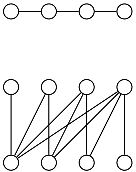

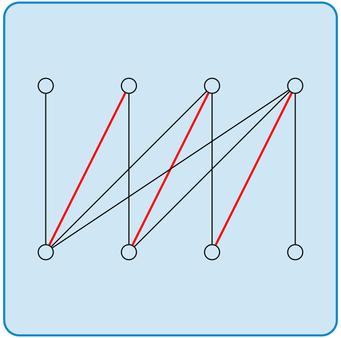

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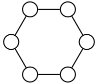

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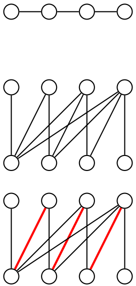

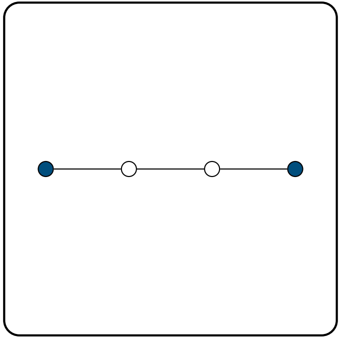

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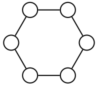

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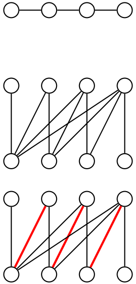

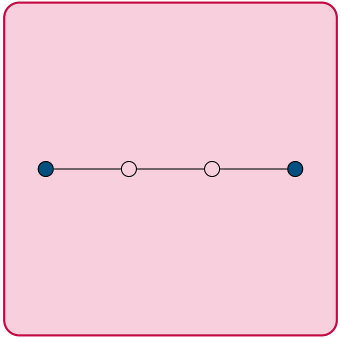

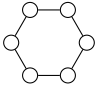

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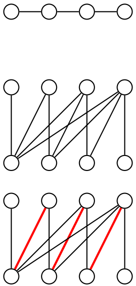

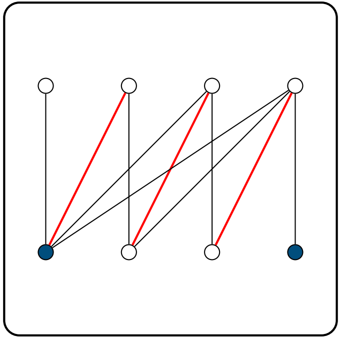

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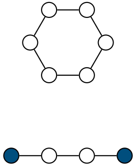

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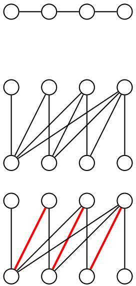

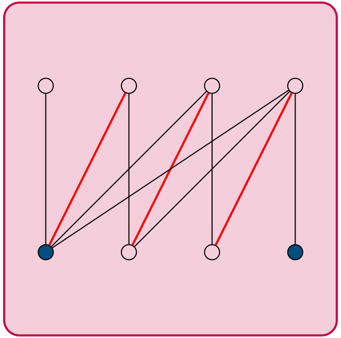

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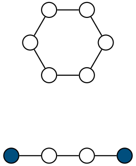

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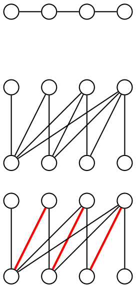

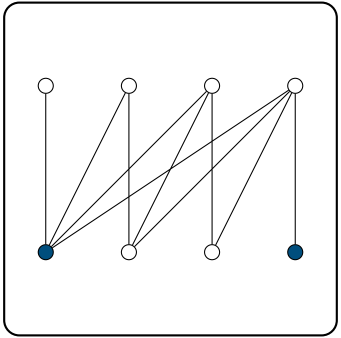

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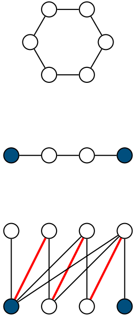

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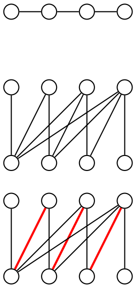

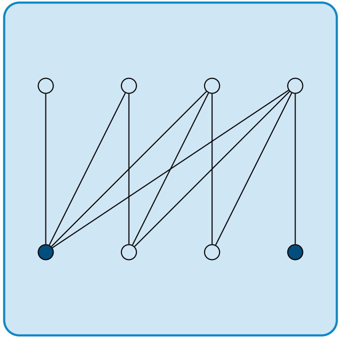

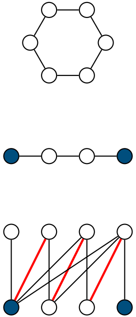

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Theorems [Ding'92, folklore]

- Bounded tree-depth $\Longrightarrow$ WQO
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For **colored paths** (i.e. finite words), WQO is decidable

- for *regular languages* [Atminas, Lozin, Moshkov, '17]
- for languages recognized by *amalgamation systems* (CFG, VASS, etc) and infixes of *morphic words* [Lhote, L, Schütze, arxiv 2025]

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Instead : consider *hereditary classes* and *freely colour* them

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$\mathsf{Label}_X(\mathcal{C})$ "X-labelled elements of $\mathcal{C}$"

$\mathcal{C}$ is $(X, \leq)$-WQO $\iff$ $\mathsf{Label}_X(\mathcal{C})$ is WQO

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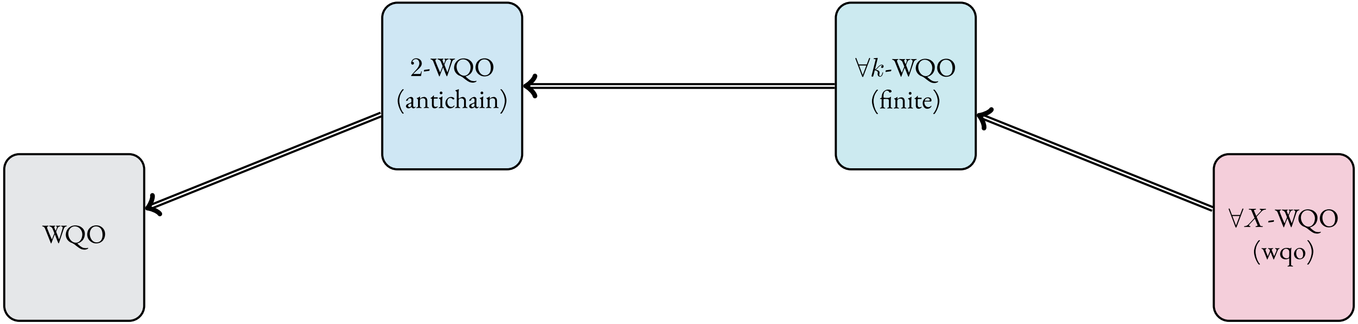

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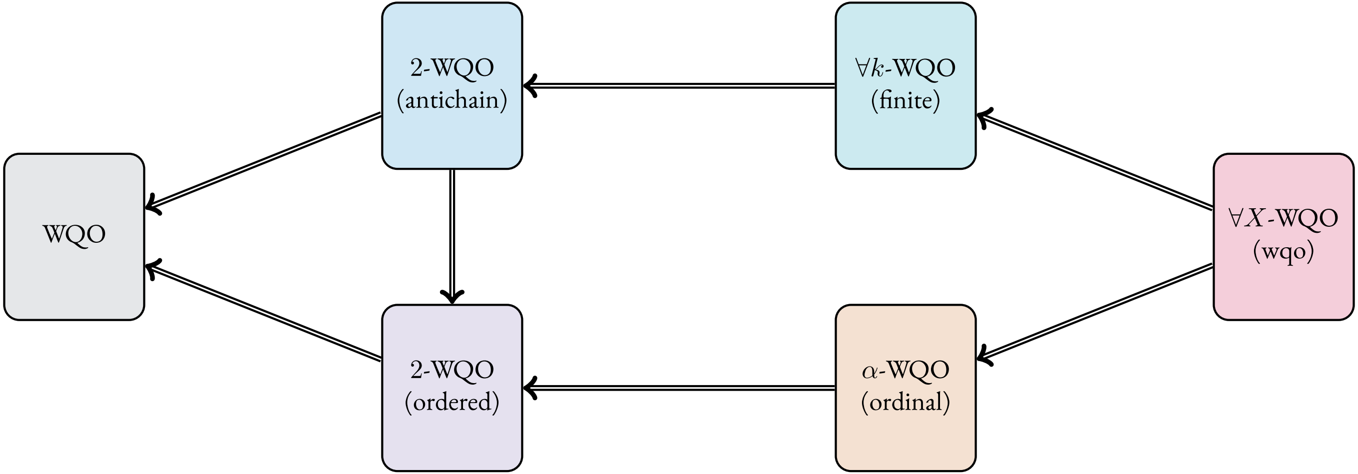

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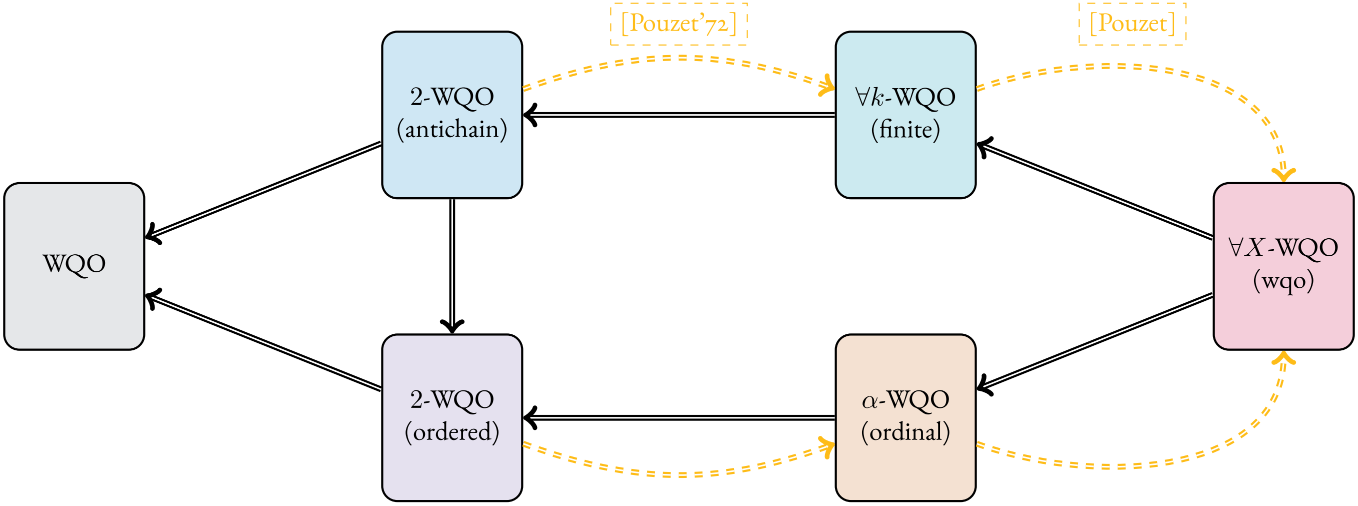

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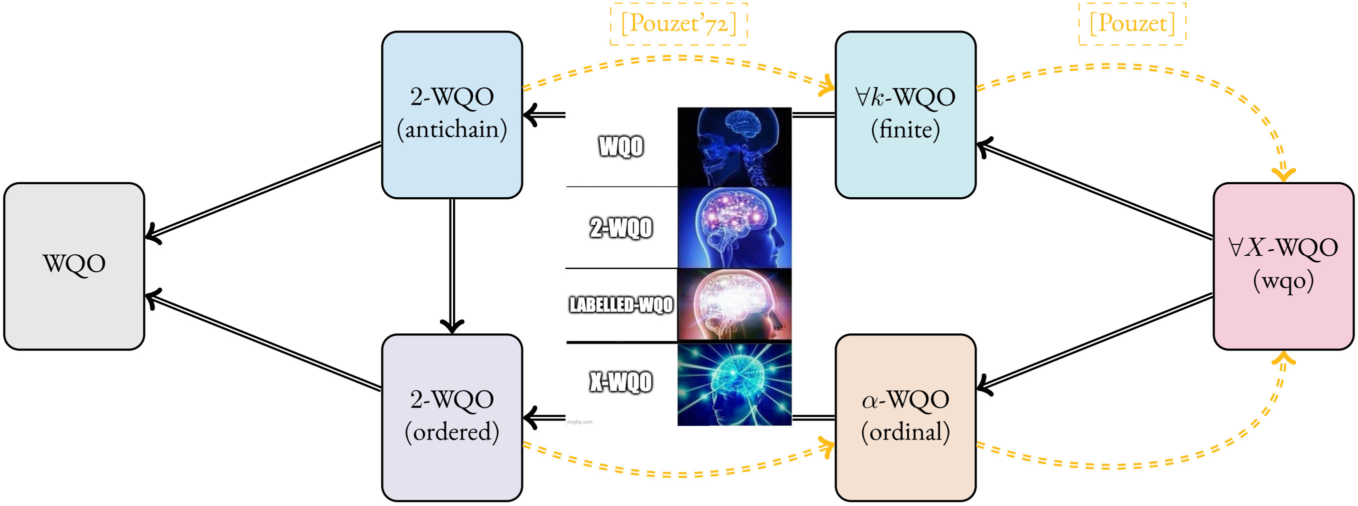
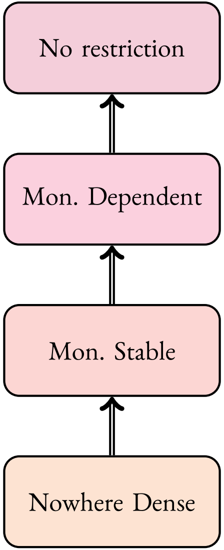


Relationship with structural graph theory



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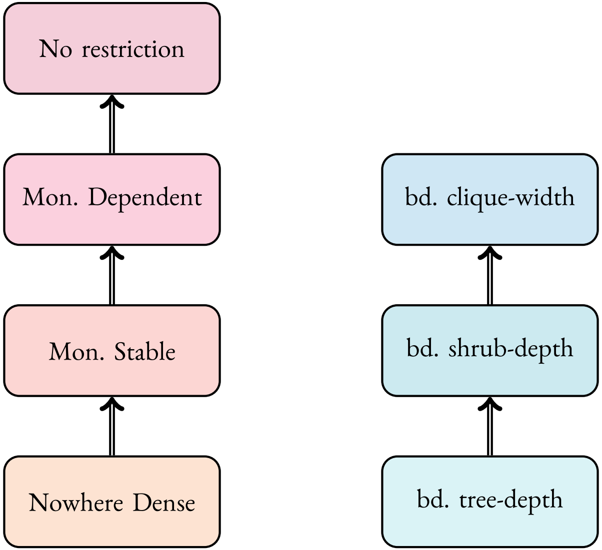

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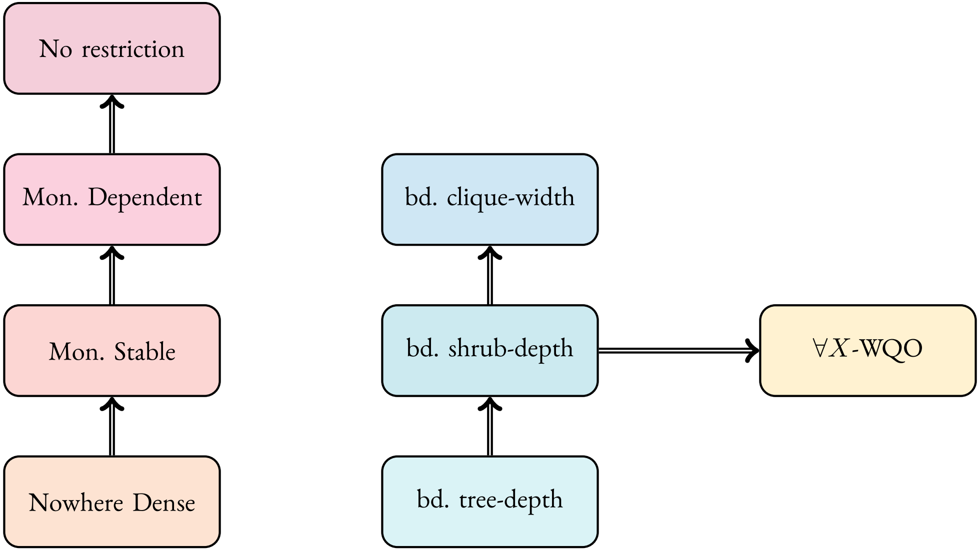

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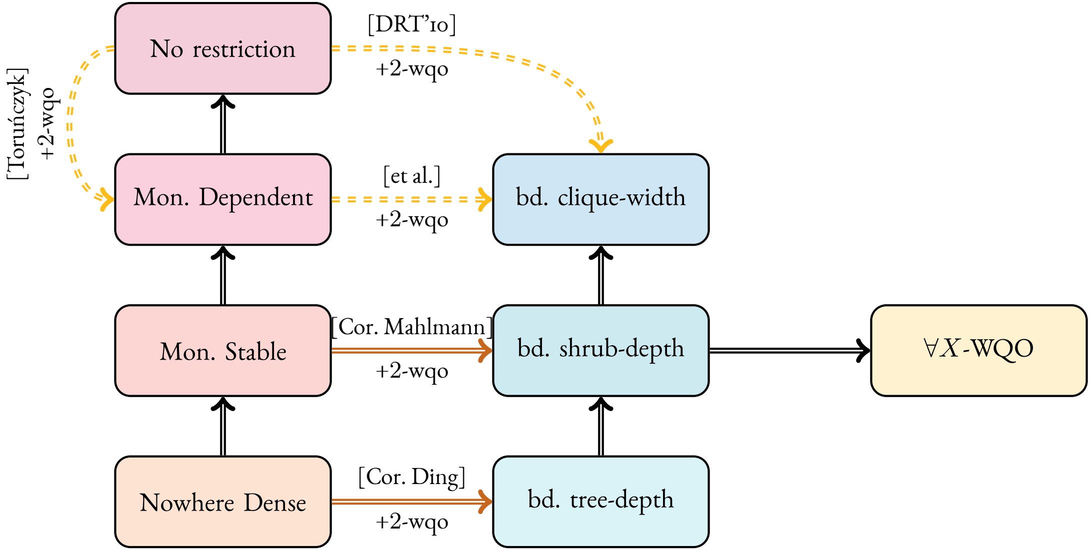

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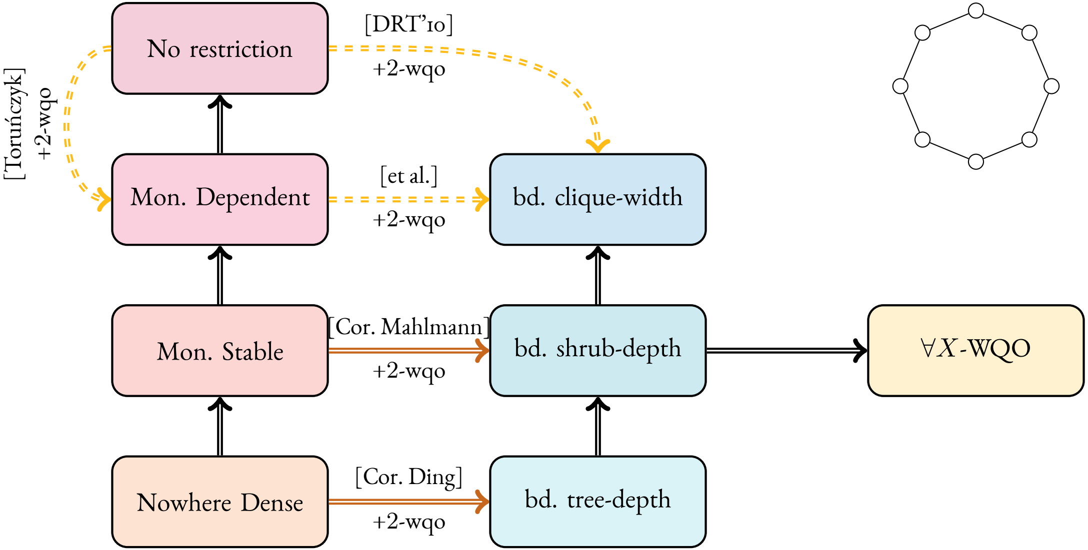

Relabel Functions / Clique-Width / NLC_ℓ

Finite Set : $Q = \{a, b, c\}$

Relabels : $\mathcal{F} = \{\mathrm{id}_Q, \rho, \rho^2\}$

$$\rho : \begin{cases} a & \mapsto b \\ b & \mapsto c \\ c & \mapsto c \end{cases}$$

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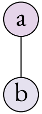

Connect : ('a', 'b')

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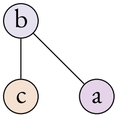

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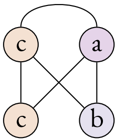

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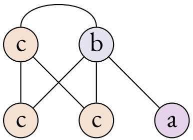

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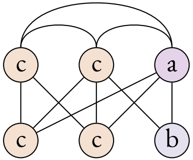

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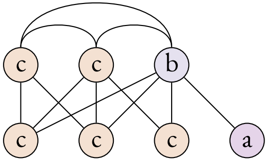

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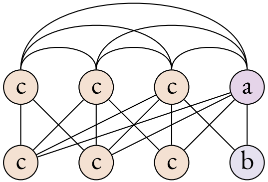

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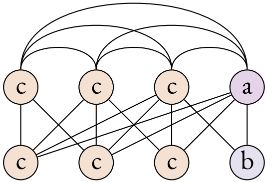
Relabels : $\mathcal{F} = \{\mathrm{id}_Q, \rho, \rho^2\}$

$$\rho : \begin{cases} a & \mapsto b \\ b & \mapsto c \\ c & \mapsto c \end{cases}$$



Connect : ('a', 'b')

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Theorem [DRT'10] One can decide whether $\mathsf{Relab}(\mathcal{F})$ is 2-wqo.
2-wqo $\iff$ $\forall X$-WQO holds on these classes.

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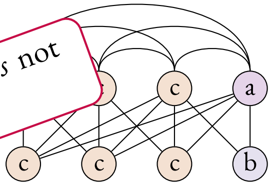

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Problem : this is a theorem on *semigroups* not *languages.*

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Accepting condition : $P \subseteq M^3$

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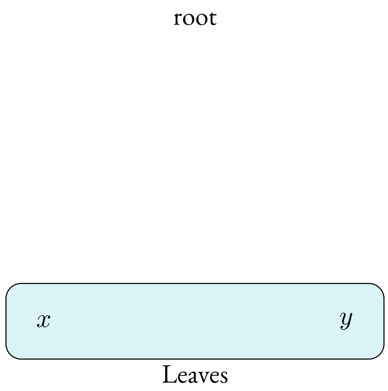

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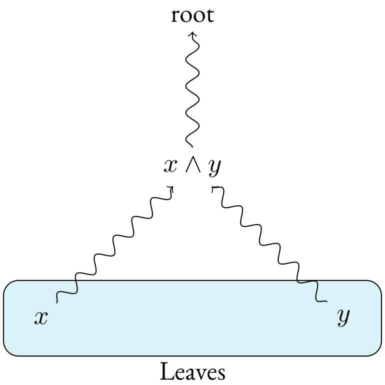

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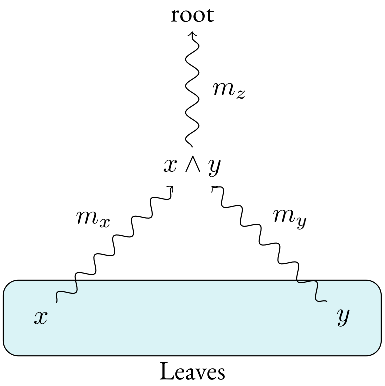

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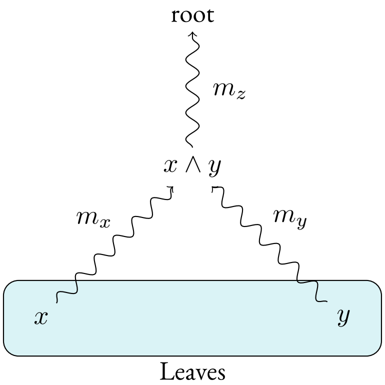

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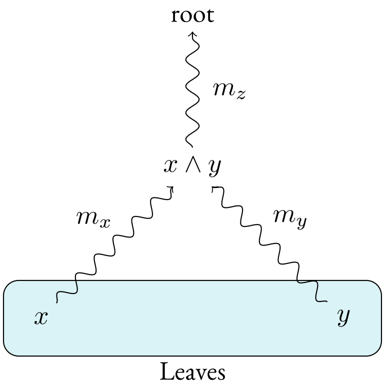

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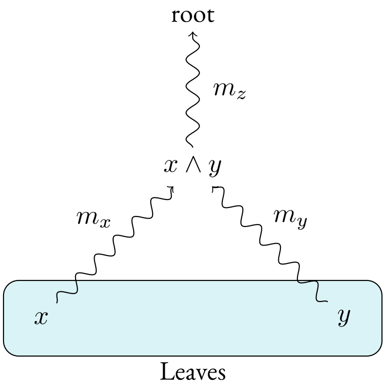

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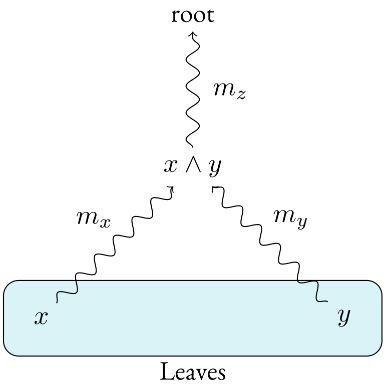
Bonus

- reduces to classes of bounded *linear* clique width
- existential transductions of finite paths



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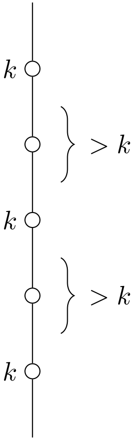

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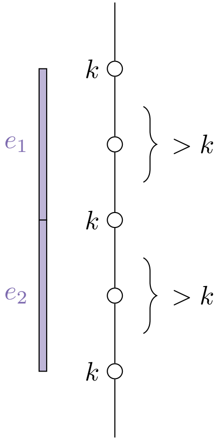

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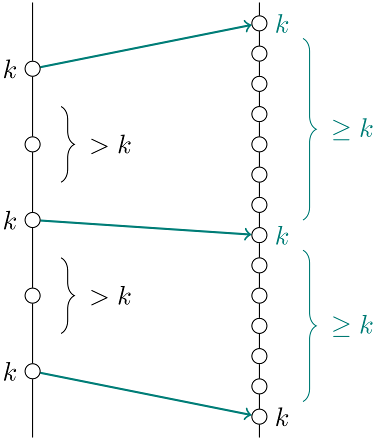

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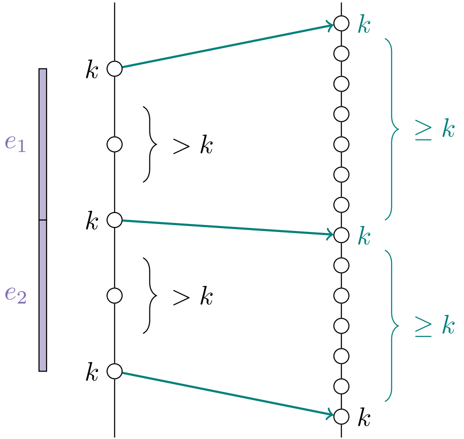

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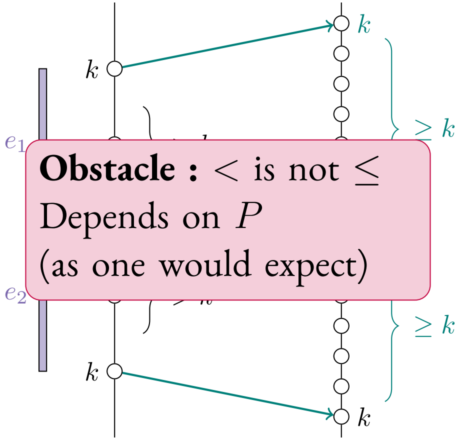

Obstacle : $<$ is not $\leq$
Depends on P
(as one would expect)

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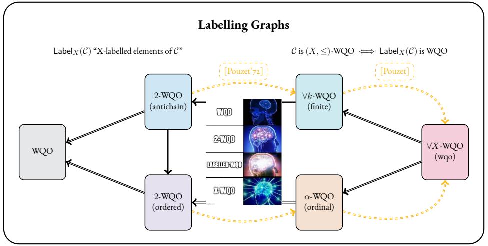
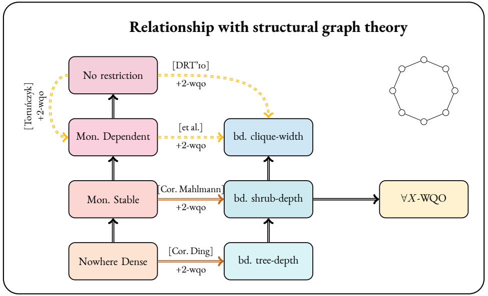
What's next ?





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2-**WQO** :
- 2-WQO (antichain) $\implies$ $\forall k$-WQO
- 2-WQO (order) $\implies$ $\forall k$-WQO
- Relationship to monadically dependent classes
- "Successor-free graphs"

WQO :
- Decide WQO for relabel functions
- Decide WQO given excluded patterns