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Abstract of the dissertation

This document is the abstract of the dissertation submitted by Aliaume Lopez for the E. W. Beth Outstanding Dissertation Prize 2024. It is organised as follows: **Introduction:** Details of the thesis, a short summary of the thesis, and a quick overview of the contributions. **Context:** Recalling the context of the thesis and the main challenges addressed. **Contributions:** A selection of main contributions of the thesis, organized by chapters. **Outlook:** A discussion of the potential impact of the thesis and future work.

1 Introduction

The thesis is titled "First Order Preservation Theorems in Finite Model Theory: Locality, Topology, and Limit Constructions". It has been pursued under the supervision of Jean Goubault-Larrecq and Sylvain Schmitz at the Laboratoire Méthodes Formelles (LMF) of ENS Paris-Saclay and the Institut de Recherche en Informatique Fondamentale (IRIF) of the Université Paris-Cité.

This thesis is built around three published articles [Lop21; Lop22; Lop23b], and other contributions that do not fit the narrative presented here are left in the bibliography of this document as *uncited personal references*. References to specific parts of the thesis have the form [Lop23a, Block Number, page page-number]. Beware that because the first pages of the thesis are *not numbered*, page numbering has a delta of 10 between the page number written on the pdf, and the *index of the page* in a PDF viewer. The page number used is the one *written in the PDF*.

1.1 General Themes

The research presented in this thesis lies at the intersection of several fields of theoretical computer science, such as *finite model theory*, *database theory*, *graph combinatorics* and *topology*. These relatively abstract fields (closer to mathematics than to engineering) find applications in computer science under the umbrella of *Formal Methods*. Under this methodology, abstractions of computer systems are studied using mathematical tools in order to prove their correctness and guarantee their safety. These applications are not marginal, and have proven to be useful in large-scale industrial settings (Amazon, Airbus, Arm, AdaCore, European Space Agency, and many others). Under this perspective of *Formal Methods*, the thesis is concerned with the following properties of programs/systems: **Termination**: Does the program/system always stop? **Expressiveness**: Can two systems compute different properties? **Optimisation:** Can the program/system be optimised?

The object of study of the thesis is the *expressiveness* of first order logic, through the lens of so-called *preservation theorems*, which are statements of the prototypical form "if a sentence φ defines a *positive property*, then it can be rewritten *without negations*". Here, the notion of *positive property* and *without negations* are left intentionally vague, as various choices lead to different theorems. As we will demonstrate, these theorems are of interest in computer science and contain in essence a termination property, that we will be expressed using the mathematical notion of *compactness* in topological spaces.

1.2 Short Abstract

A majority of this thesis is dedicated to the study of preservation theorems in first-order logic and their relativisation to classes of finite structures. These theorems are classical results obtained in the second half of the 20th century [CK90], and are motivated by their role in connecting the syntactic properties of first order theories and their semantic properties, the latter being understood as properties of their collection of (potentially infinite) models. Typical questions are of the form “which classes of models can be axiomatised by universal sentences”, or “which classes of models can be axiomatised by positive existential sentences”, etc. In this document, we will focus on the restricted case where the axiomatisation uses finitely many sentences, which is equivalent as using exactly one sentence in first order logic.

Besides the theoretical interest of such questions in terms of expressiveness, these theorems actually have practical a practical impact in computer science, where they can be used to characterise *syntactic classes* of database queries for which the termination and correctness of database algorithms are guaranteed [see, e.g. Lib11].

Studying these theorems becomes notably challenging when focussing on finite models, which is precisely when they become relevant in computer science, because databases are usually modelled as *finite objects*. As such, the study of the *relativisation* preservation theorems to classes of finite structures has dual motivations. On the one hand, it contributes to the ongoing effort of developing a *Finite Model Theory*, where they can play the role of classical theorems that can be obtained in the finite by *combinatorial* methods, that vastly differ from the original proofs [see the discussions of Ros95; and Ros08]. On the other hand, their relativisation (or lack thereof) to classes of finite structures characterises termination and correctness of algorithms in database theory [DNR08; Lib11].

There is up to this date, no clear understanding of the relativisation of preservation theorems to classes of (finite) structures. Identifying such “well-behaved” classes (in the terminology of Atserias, Dawar, and Grohe [ADG08]) has been an active domain of research since the 60s, with a series of negative results [CF21; Kup21; DS21; Sto95; AG94; AG87; Tai59] interleaved with positive ones [BC19; SAC16; SAC14; DRT10; Ros08; ADG08; ADK06; Din92]. From this non exhaustive list of positive and negative results, the goal is to understand that the *relativisation* of a given preservation theorem is a non trivial property of classes of structures, and that it does not seem to follow (easily) predictable patterns.

Let us illustrate one of the difficulties by considering one example of a preservation theorem, the *Łoś-Tarski theorem* [Łoś55; Tar54]. This theorem relativises to the class of all *finite cycles*, relativises to the class of all *finite paths*, but does not relativise to their their union, the class of *finite cycles or finite paths* [Lop23a, Example 5.1.11.]. Lacking composition properties is an issue both for the theoretical study of such classes, and for the practical applications: slight modifications of the class of structures can lead to a complete change in behaviour.

This thesis presents a systematic approach to investigating preservation theorems in Finite Model Theory. The motivation is to produce tools and theorems that are able to explain *when* and *why* some classes of structures are well-behaved with respect to preservation theorems. This approach is based on two main ideas: first, consider only first order logic [as opposed to Ros95, for instance], and second, interpret these theorems as *topological* properties of classes of finite structures. These two ideas respectively allow to generalise the techniques based on locality (a central tool in Finite Model Theory), and to provide a compositional theory for preservation theorems (which was previously lacking).

Finally, the topological presentation of preservation theorems introduced in this thesis is closely related to the notion of *Noetherian Space* that was used by Goubault-Larrecq [Gou07] to provide algorithm for the verification of infinite state transition systems, fully developed in [Gou10]. In this specific setting, a fixed point theorem is obtained that allows the *inductive* definition of *Noetherian topologies*. This result has its own interest in the verification of infinite state systems, but can also be interpreted as a first step towards studying preservation theorems on *inductively defined* classes of structures.

1.3 Preservation Theorems and Topology

In order to present and illustrate the main results obtained in this thesis, we will need to introduce some technical notions that may not be familiar to the reader. The first one is the notion of *preservation theorem* in the context of finite model theory, and the second one is the notion of *topological space*.

What is a preservation theorem? In the thesis, a very general form of preservation theorem is considered, which requires the introduction of *morphisms* given a fragment F of first order logic [see Lop23a, Definition 3.1.1, page 30]. A map $h: A \to B$ between two models is a morphism for F when, for every formula $\psi \in \mathsf{F}$, for every tuple $\vec{a}$ that satisfies ψ in A, the image of $\vec{a}$ through h satisfies ψ in B. As an example, the morphisms for the fragment of positive quantifier free formulas are exactly the usual *homomorphisms*. As another example, when $\mathsf{F} = \mathsf{FO}$, the morphisms are exactly the *elementary embeddings* of traditional model theory.

In this setting, the F-*preservation theorem* states that the following are equivalent for every first order sentence φ [restated in Lop23a, Theorem 3.1.9, page 33]:

1. φ is **preserved under** F, i.e., if $A \models \varphi$, and $h: A \to B$ is an F-morphism, then $B \models \varphi$.
2. φ is equivalent to some sentence in $\exists\mathsf{F}$.

As examples of preservation theorem, the Łoś-Tarski theorem considers F to be quantifier free formulas [Łoś55; Tar54], and the quantifier free morphisms are called *embeddings*. Other examples include preservation under homomorphisms or Lyndon's positivity theorem [Lyn59]. This synthetic presentation of preservation theorems can already be seen as some kind of contribution, because it proposes a formal statement for a previously informal collection of similarly shaped theorems.

How to prove relativisation? There is one key argument that is behind most of the proofs of relativisation to classes of finite structures, and states that it is equivalent to find an equivalent sentence in $\exists\mathsf{F}$ (a syntactic property) or to prove that the sentence has *finitely many minimal models* (with respect to F-*morphisms*). This remark holds for the Łoś-Tarski theorem, as well as for the homomorphism preservation theorem, the two examples that will be used in this document [see Lop23a, Lemma 3.2.3, page 38].

Then, there are essentially two (non-exclusive) approaches to proving that sentences do not have infinitely many minimal models. The first one is to leverage the *combinatorial* properties of the class, the extreme example being when the class $\mathcal{C}$ itself is finite. The second one is to leverage the properties of the *logic*, which can take the form of Ehrenfeucht–Fraïssé games or using the Gaifman locality theorem [Gai82].

As an example of the combinatorial approach, Ding [Din92] proves that the Łoś-Tarski theorem relativises to classes $\mathcal{C}$ of structures that have *bounded tree-depth* [restated in Lop23a, Lemma 3.3.10, page 45], without even mentioning logic.[1] As an example of the logical approach, classical locality arguments stating that a fixed sentence φ can only distinguish finitely many cycles incidentally demonstrate that the Łoś-Tarski theorem relativises to the class of all finite cycles [restated in Lop23a, Example 4.1.1, page 73].

Why topology? A first incentive to use topological tools to explain preservation theorems is that their *proof* in the classical case rely on the so-called compactness theorem of first order logic, which is, essentially, a topological theorem about the *compactness* of some space. While this might not seem very helpful, let us also argue that *compactness* in topological spaces is a notion meant to generalise *finiteness*, which is precisely what happens in the proof scheme described in How to prove relativisation? Informally, in a topological space, a set is *compact* if

[1] It leverages a notion called *well-quasi-orderings*, that will appear later on in this document, but will never be defined [we refer Dem+12, for a reasonably complete survey on the topic].

it behaves as a finite set with respect to the topology. To be more precise, in a topological space (X, τ) a set E is *compact* if, for every sequence $(U_i)_{i\in I}$ such that $E \subseteq \bigcup_{i\in I} U_i$, there exists a finite subset $J \subseteq_{\mathsf{fin}} I$ such that $E \subseteq \bigcup_{i\in J} U_i$. Notice that every *finite set* is compact.

For instance, in a proof of relativisation of the Łoś-Tarski theorem to a hereditary class $\mathcal{C}$ of finite structures, the collection of models of a given sentence φ *preserved under extensions* is not going to be finite *per se*. However, it will *behave as a finite set*, because it can be described using finitely many minimal models. Formally, this is done by considering the topology of *upwards closed* subsets of $\mathcal{C}$, and noticing that the collection of models of φ is *compact* if and only if it has *finitely many minimal models*.

Another argument in favor of a topological approach is the robustness of the recently developed of a theory of *Noetherian spaces* for the verification of infinite state transition systems started by Goubault-Larrecq [Gou07; Gou10; Gou13; Gou22a; GHL22; GL23]. These generalise the notion of *well-quasi-ordering*, which is a key combinatorial concept in computer science, and *implies* preservation theorems. In a sense that will be made precise, the *Noetherian spaces* are a *logic-less* form of preservation theorems [see Lop23a, Section 6.2.1, page 146]. To build a comprehension of which classes of structures enjoy preservation theorems, it is therefore natural to develop a theory that encompasses those obtained via Noetherian spaces.

2 Contributions

In this context, the thesis proposes the following main contributions that we group thematically. For each theme, two results (definitions or theorems) are chosen to illustrate the approach taken together with their outcomes. We will specifically refer to pages in the thesis [Lop23a] to allow the reader to find more details and specific statements if needed.

2.1 Locality Based Approach [Lop23a, Chapters 4 and 5]

The first two contributions are focused on proving that a specific preservation theorem relativises, namely, the Łoś-Tarski preservation theorem. Recall that this theorem states the equi-expresiveness of *existential sentences* and *sentences preserved under extensions*.

A first contribution is the introduction and characterization of a *positive* variant of the Gaifman locality theorem [Gai82], that is at the core of most of the recent proofs of relativisation for preservation theorems [see e.g. ADK06; ADG08; Daw10]. The rationale behind this result is that the difficult implication of a preservation theorem is to transform a semantic property into a syntactic one (the other direction being often a simple induction on the syntax), and that the Gaifman locality theorem provides a first syntactic decomposition of an arbitrary sentence into a Boolean combination of so-called *basic local sentences*, over which it is possible to observe some semantic properties.

Semantically, a *basic local sentence* searches for a collection of *disjoint* neighbourhoods of the structure, that all satisfy a given first order property. One of the main issues with the Gaifman locality theorem is that it introduces *negations* of the basic local sentences, which are hard to handle in the context of preservation theorems.

Having a better first decomposition, by *removing outer negations* in the Gaifman normal form takes us closer to *existential sentences*. It also puts into light the fragment of *existential local sentences*, that correspond to some intermediate ground between *arbitrary* sentences and *existential sentences* [see Lop23a, Figure 4.1, page 68]. Note that similar kind of sentences were successfully applied to automata theory by Schwentick and Barthelmann [SB99].

> **Positive Locality Theorem [Lop23a, Theorem 4.2.2, page 75]:** existential local sentences express the same properties as positive Boolean combination of basic local sentences.

This theorem has its own interest as a new variation around Gaifman locality. However, it is distinguished because it provides an intuition on the proof schemes developed when studying relativisation properties: half of the work is often to remove negations from a Gaifman normal form, which precisely corresponds to building an existential local sentence. Only then, in a second proof, one moves from an existential local sentence to a sentence of the desired fragment.

Following the ideas of considering the local behaviours of the structures, and noticing that *existential local sentences* over a class $\mathcal{C}$ of structures correspond to *existential sentences* over the *local neighbourhoods* of structures in $\mathcal{C}$, we obtained a powerful **local-to-global** preservation theorem, that subsumes results obtained in the literature [Lop23a, Theorem 5.1.5, page 121], and fully characterises the Łoś-Tarski preservation theorem as a *local* property (under classical assumptions).

> **Local-to-Global Theorem [Lop23a, Theorem 5.1.2, page 120]:** the Łoś-Tarski theorem relativises to a class $\mathcal{C}$ of structures if and only if it relativises *locally* to $\mathcal{C}$ (under the assumption that $\mathcal{C}$ contains only finite structures, is hereditary, and is closed under disjoint unions).

One measure of the importance of this result is that it provides new families of properties for which the Łoś-Tarski theorem relativises, obtained by *localising* previously known properties. This is illustrated in [Lop23a, Figure 5.2, page 122]. As a nice side effect, we recover the main result of Atserias, Dawar, and Grohe [ADG08] by noticing that the Łoś-Tarski theorem relativises to any *finite class* of structures (which is a folklore result), and noticing that the notion of *well-behaved* used by Atserias, Dawar, and Grohe [ADG08] amounts to being *locally finite* in our terminology [Lop23a, Exercise 5.1.4, page 121]. The importance of this result can also be seen as a way to *decouple* the study of preservation theorems into a *logical part* (the local-to-global theorem) and a *combinatorial part* (the structural behaviours of local neighbourhoods).

2.2 Topological Approach [Lop23a, Chapter 6]

During this thesis, a focus has been taken on figuring out what *topological properties* correspond to preservation theorems, originally motivated by the resemblance between the relativisation of *preservation theorems* and *Noetherian spaces*. The main motivation for this abstract study is to find a way around the non-compositionality of preservation theorems and the complexity of proving their relativisation. In the topological setting Noetherian spaces have proven to be quite robust and versatile [see for instance the dedicated chapter of Gou13].

This sparked the study of triples $(X, \tau, \mathcal{B})$, where (X, τ) is a topological space, and $\mathcal{B}$ is a Boolean subalgebra of $\mathcal{P}(X)$, called *logically presented pre-spectral spaces* (LPPS). Note that the name *pre-spectral* comes from the theory of *spectral spaces* [DST19], also known as *Priestley spaces* [GG24], that play a key role in the duality between logic (or lattices) and topological spaces, dating back to the works of Johnstone [Joh82].

> **LPPS Definition [Lop23a, Definition 6.1.11, page 143]:** a logically presented pre-spectral space is a triple $(X, \tau, \mathcal{B})$, where τ captures the *open properties* of the space, $\mathcal{B}$ captures the *definable properties* of the space, in such a way that: *definable properties* generate the *open properties*, and properties that are simultaneously *open* and *definable* are *compact*.

This definition is distinguished because it provides a new way to think about preservation theorems in terms of topological compactness, in a way that connects them more clearly to Noetherian spaces, but also to more exotic spaces, such as spectral spaces. Furthermore, it abstracts the considered logic(s) as a Boolean subalgebra of the class of structures considered, opening the door to the study of non-FO based preservation theorems, without any change in definitions.

The main results of this approach are that *lpps* characterises a relatively large class of relativisation of preservation theorems [Lop23a, Theorem 6.1.12, page 143]. And that this abstract definition actually allows to **compose** preservation theorems to obtain new preservation theorems from known ones in a systematic fashion. The archetypal example of such composition theorem is the *composition by substitution* theorem restated below.

> **Substitution Theorem [Lop23a, Theorem 6.3.44, page 169]:** Let $(X, \tau, \mathcal{B})$ be an **lpps**, and and $(\mathcal{C}, \leq)$ be a class of coloured structures that is **well-quasi-ordering**. Then the class of structures obtained by colouring elements of $\mathcal{C}$ by elements of X is an **lpps**.

This result is distinguished because it encompasses many results at once, and it is connected to conjectures by Pouzet. It for instance states that if $(X, \tau, \mathcal{B})$ is an **lpps**, then so are the *words* over X, similarly for the *trees* over X, which are analogues of the well-known *Higman* and *Kruskal* theorems in well-quasi-order (wqo) theory [Hig52; Kru72]. Furthermore, it provides a positive answer to the "lpps-variant" of a conjecture by Pouzet, asking whether "∞-wqo" (that is, being wqo for any colouring with finitely many colours) implies "wqo-wqo" (that is, being wqo when coloured by any wqo).

2.3 Logic-free Preservation Theorems [Lop23a, Chapters 7 and 8]

One of the main drawbacks of the approach of preservation theorems by topological means is that they do not interact well with *inductive definitions*. This may seem odd, as we have seen that *words* and *trees*, both inductively defined constructions, preserve *lpps*. However, it should become less surprising in the light of the fact that these constructions preserve well-quasi-orderings is already non-trivial [Hig52; Kru72]. In order to understand inductively defined construction, this thesis focused on a *logic-less* variant of *lpps*, that is, considering that the logic is powerful enough express **every subset**. These are precisely the Noetherian spaces, for which the study of inductively defined constructors is highly non-trivial.

Let us take a moment to discuss on the dual vision of *words* and *trees*, respectively as (first-order definable) classes of finite structures, with potential unary predicates on their nodes, and as an *inductive* construction guided by the equations $X^* := 1 + X \times X^*$ and $T(X) := 1 + X \times T(X)^*$. The notions of *subword* and *subtrees* are easily expressed in terms of *embedding* of relational structures, and are not the orderings that would arise naturally from the inductive constructions (in a categorical sense). Surprisingly, the proofs that these order relations yields *well-quasi-orderings* are highly non-trivial and do rely on the inductive constructions of *words* and *trees* [Hig52; Kru72].

The main contribution is the introduction of *topology expanders* as "well-behaved topology constructors." This definition is distinguished because it is an abstraction that captures all the examples and non-examples of limit constructions in Noetherian spaces that I know of.[2] Furthermore, the interesting part of the definition is side-stepping from the study of inductive construction of spaces, to inductive constructions of a topology over a given (fixed) space. This change of perspective is what makes the definition sound, and gives such a flexibility to the theory.

> **Topology Expander Definition [Lop23a, Definition 7.2.17, page 189]:** Given a set X, a *topology expander* is a map F from topologies over X to topologies over X, that is increasing, preserves Noetherian topologies, and is compatible with restriction to subsets.

Topology expanders have proven to be of interest by the two following theorems, that respectively validate their use to construct *limits* of spaces, and that these limits can be used to construct Noetherian topologies over *inductively defined spaces* in a way that generalises all previously known constructions.

[2] Note that every *well-quasi-ordering* is a *Noetherian space*, but the converse is not true. In particular, topology expanders will allow to construct inductively the *subword* and *subtree* relations.

> **Noetherian Limit Theorem [Lop23a, Theorem 7.2.33, page 197]:** Least fixed points of topology expanders are Noetherian topologies.
>
> **Noetherian Inductive Theorem [Lop23a, Theorem 8.2.33, page 227]:** For every inductive constructor G of Noetherian spaces, there exists a canonical *divisibility topology* over the least fixed point of G, that is *Noetherian*, and generalises strictly previous results on *well-quasi-orderings*.

Note that in the statement of this last theorem, we took the time to compare our approach to a previous attempt at devising generic fixed point theorems for *well-quasi-orderings* by Hasegawa [Has02]. These two theorems effectively close the discussion on inductive definition of Noetherian spaces: they allow to recover *all* theorems in the literature, even for non-inductive constructors (using the limit theorem in those cases), with *shorter*, *simpler* and *generic* proofs. As a side effect, we obtain that previously defined topologies, that contained a part of *tweaking*[3] were in fact "canonical" instances of the *divisibility topology*.

3 Outlook

The results provided in this thesis pave the way to several interesting research directions. While the focus on *preservation theorems* is well-motivated, applications to computer science would typically require some kind of *effectiveness*. Similarly, the *qualitative results* (compactness) often have *quantitative* counterparts (ordinal invariants) that are not studied here. A typical motivation for such effective and quantitative theories would be the verification of *database transition systems*, by analogy with *Well-Structured Transition Systems* based on *well-quasi-orderings* or *Noetherian spaces* [Abd+96; Gou10]. Furthermore, this thesis focuses on *general methods* for obtaining relativisation of preservation theorems. However, another approach could be to devise *algorithms* that, given a representation of a class $\mathcal{C}$ of structures, decide whether they are suitable for a preservation theorem. This could take the form of results like Daligault, Rao, and Thomassé [DRT10], where the property of being *well-quasi-ordered* for the extension relation is proven to be decidable.[4]

In a more theoretical research direction, the *Inductive Noetherian Theorem* calls for a similar *Inductive LPPS Theorem*. The latter could provide another viewpoint on the (finite) *Homomorphism Preservation Theorem* of Rossman [Ros08]. Indeed, this result is an outlier by its nature (most preservation theorems do not relativise to all finite structures) and its proof that develops an *ad-hoc* saturation property using existential types. Casting this result in the form of a *limit* property would provide a better understanding of its combinatorial nature, and potentially allow to devise *variations* of this result. Note that for now, the only variations and improvements known are obtained by either using the theorem as a black box [NO12; BC19], or by the author himself using deep results from circuit complexity [Ros16].

Finally, the characterization of the Łoś-Tarski theorem as a *local property* is unsatisfactory for classes where *local behaviours* are as complex as *global ones*. For instance, the class of all cliques is as complex as the class of *local neighbourhoods* of cliques, as the two are in fact equal. This begs for better decompositions of classes of structures, and a *local-to-global* theorem adapted to these decompositions. Inspiration can be found in the ongoing effort to characterize *efficient query answering* on classes of structures by combining *locality* with the notion of *flip* (reversing edges and non-edges in a graph): for instance, the flip of a clique is an independent set, for which locality based technique work amazingly well [see e.g. Gaj+23].

[3] When choosing a topology for a space, there is a trade-off between adding a lot of expressiveness (open subsets) and proving that the topology is *Noetherian* (not too many open subsets). In general, there is not *one* topology that is Noetherian and contains as many opens as possible: it is always possible to add *some* new open subsets, and remain Noetherian.

[4] In this paper, the decision problem is not solved for a fixed class of structures, but could probably be adapted to this case. Furthermore, they focus on *graphs*, but this could be generalised to relational structures in a straightforward way.

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